# Which Free Lunch Would You Like Today, Sir?

## Delta hedging, volatility arbitrage and optimal portfolios

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**In this lecture. . .**

- Volatility arbitrage
- How to hedge
- Expected profit
- Variance of profit
- Optimizing a portfolio

## Introduction

In this lecture we examine the profit to be made hedging options that are mispriced by the market.

- This is the subject of how to delta hedge when your estimate of future actual volatility differs from that of the market as measured by the implied volatility (Natenberg, 1994).

Since there are two volatilities in this problem, implied and actual, we have to study the effects of using each of these in the classical delta formula (Black & Scholes, 1973).

We'll see how you can hedge using a delta based on *either* actual volatility or on implied volatility.

- Neither is wrong, they just have different risk/return profiles.

Part of what follows repeats the excellent work of Carr (2005) and Henrard (2001). Carr derived the expression for profit from hedging using different volatilities. Henrard independently derived these results and also performed simulations to examine the statistical properties of the possibly path-dependent profit. He also made important observations on portfolios of options and on the role of the asset's growth rate in determining the profit.

This lecture extends their analyses in several directions.

Some of the work in this lecture has been used successfully by a volatility arbitrage hedge fund.

## Overview on the lecture

The lecture will consist of several main parts:

- First we set up the problem by explaining the role that volatility plays in hedging.

- We look at the mark-to-market profit and the final profit when hedging using actual volatility.

- We then examine the mark-to-market and total profit made when hedging using implied volatility. This profit is path dependent. We briefly repeat the analyses of Carr (2005) and Henrard (2001).

Towards the end of the lecture we focus on the latter case of hedging using implied volatility, which is the more common market practice.

- Because the final profit depends on the path taken by the asset in this case we look at simple statistical properties of this profit. We derive a formula for the expected total profit and a formula for the variance of this profit.

- We then consider portfolios of options, and again find closed-form formulas for the expectation and variance of profit.

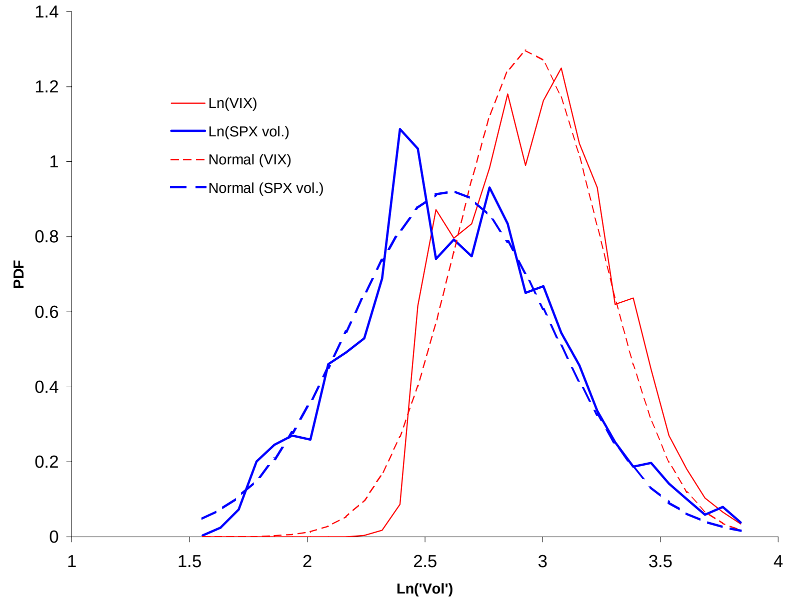
- To find the full probability distribution of total profit we could perform simulations (Henrard, 2001) or solve a three-dimensional differential equation. We outline the latter approach. This is to be preferred generally since it will be faster than simulations, therefore making portfolio optimizations easier to perform.

- Next we outline a portfolio selection method based on exponential utility.

- Finally, we consider portfolios of options on many underlyings.

Technical details are contained at the end.

## Before we start...




Distributions of the logarithms of the VIX and the rolling 30-day realized SPX volatility, and the normal distributions for comparison.

Above is a simple plot of the distributions of the logarithms of the VIX and of the rolling 30-day realized SPX volatility using data from 1990 to mid 2005. The VIX is an implied volatility measure based on the SPX index and so you would expect it and the realized SPX volatility to bear close resemblance.

However, as can be seen in the figure, the implied volatility VIX seems to be higher than the realized volatility.

Both of these volatilities are approximately lognormally distributed (since their logarithms appear to be Gaussian), especially the realized volatility. The VIX distribution is somewhat truncated on the left.

The mean of the realized volatility, about 15%, is significantly lower than the mean of the VIX, about 20%, but its standard deviation is greater.

## Implied versus actual, delta hedging but which volatility?

• Actual volatility is the amount of 'noise' in the stock price, it is the coefficient of the Wiener process in the stock returns model, it is the amount of randomness that 'actually' transpires.

• Implied volatility is how the market is pricing the option currently. Since the market does not have perfect knowledge about the future these two numbers can and will be different.

Imagine that we have a forecast for volatility over the remaining life of an option, this volatility is forecast to be constant, and further assume that our forecast turns out to be correct.

Let's buy an underpriced option and delta hedge to expiry. *But which delta do you choose?* Delta based on actual or implied volatility?

Scenario: Implied volatility for an option is 20%, but we believe that actual volatility is 30%.

Question: How can we make money if our forecast is correct?

Answer: Buy the option and delta hedge.

But which delta do we use?

We know that

$$\Delta = N(d_1)$$

where

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{s^2}{2}} ds$$

and

$$d_1 = \frac{\ln(S/E) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}.$$

We can all agree on $S$, $E$, $T-t$ and $r$ (almost), but not on $\sigma$. So should we use $\sigma = 0.2$ or $0.3$?

In what follows we use $\sigma$ to denote actual volatility and $\tilde{\sigma}$ to represent implied volatility, both assumed constant.

## Case 1: Hedge with actual volatility, $\sigma$

By hedging with actual volatility we are replicating a short position in a *correctly priced* option.

The payoffs for our long option and our short replicated option will exactly cancel.

The profit we make will be exactly the difference in the Black–Scholes prices of an option with 30% volatility and one with 20% volatility.

If $V(S,t;\sigma)$ is the Black–Scholes formula then the guaranteed profit is

$$V(S,t;\sigma) - V(S,t;\tilde{\sigma}).$$

But how is this guaranteed profit realized? Let's do the math on a mark-to-market basis.

In the following, superscript '$a$' means actual and '$i$' means implied, these can be applied to deltas and option values. For example, $\Delta^a$ is the delta using the actual volatility in the formula. $V^i$ is the theoretical option value using the implied volatility in the formula. Note also that $V$, $\Delta$, $\Gamma$ and $\Theta$ are all simple, known, Black–Scholes formulas.

The model is the classical

$$dS = \mu S \, dt + \sigma S \, dX.$$

Now, set up a portfolio by buying the option for $V^i$ and hedge with $\Delta^a$ of the stock.

The values of each of the components of our portfolio are shown in the following table.

| **Component** | **Value** |
|---|---|
| Option | $V^i$ |
| Stock | $-\Delta^a\, S$ |
| Cash | $-V^i + \Delta^a\, S$ |

Portfolio composition and values, today.

Leave this hedged portfolio overnight, and come back to it the next day. The new values are shown in the next table. (We have included a continuous dividend yield in this.)

| **Component** | **Value** |
|---|---|
| Option | $V^i + dV^i$ |
| Stock | $-\Delta^a\, S - \Delta^a\, dS$ |
| Cash | $(-V^i + \Delta^a\, S)(1 + r\, dt) - \Delta^a DS\, dt$ |

Portfolio composition and values, tomorrow.

Therefore we have made, mark to market,

$$dV^i - \Delta^a\, dS - r(V^i - \Delta^a\, S)\, dt - \Delta^a DS\, dt.$$

Because the option would be correctly valued at $V^a$ then we have

$$dV^a - \Delta^a\, dS - r(V^a - \Delta^a\, S)\, dt - \Delta^a DS\, dt = 0.$$

So we can write the mark-to-market profit over one time step as

$$dV^i - dV^a + r(V^a - \Delta^a\, S)\, dt - r(V^i - \Delta^a\, S)\, dt$$

$$= dV^i - dV^a - r(V^i - V^a)\, dt = e^{rt}\, d\left(e^{-rt}(V^i - V^a)\right).$$

That is the profit from time $t$ to $t + dt$. The present value of this profit at time $t_0$ is

$$e^{-r(t-t_0)} e^{rt}\, d\left(e^{-rt}(V^i - V^a)\right) = e^{rt_0}\, d\left(e^{-rt}(V^i - V^a)\right).$$

So the total profit from $t_0$ to expiration is

$$e^{rt_0} \int_{t_0}^{T} d\left(e^{-rt}(V^i - V^a)\right) = V^a - V^i.$$

This confirms what we said earlier about the guaranteed total profit by expiration.

We can also write that one time step mark-to-market profit (using Itô's lemma) as

$$\Theta^i\,dt + \Delta^i\,dS + \tfrac{1}{2}\sigma^2 S^2 \Gamma^i\,dt - \Delta^a\,dS - r(V^i - \Delta^a S)\,dt - \Delta^a DS\,dt$$

$$= \Theta^i\,dt + \mu S(\Delta^i - \Delta^a)\,dt + \tfrac{1}{2}\sigma^2 S^2 \Gamma^i\,dt$$

$$-r(V^i - V^a)\,dt + (\Delta^i - \Delta^a)\sigma S\,dX - \Delta^a DS\,dt$$

$$= (\Delta^i - \Delta^a)\sigma S\,dX + (\mu + D)S(\Delta^i - \Delta^a)\,dt + \tfrac{1}{2}\left(\sigma^2 - \tilde{\sigma}^2\right) S^2 \Gamma^i\,dt$$

(using Black–Scholes with $\sigma = \tilde{\sigma}$)

$$= \tfrac{1}{2}\left(\sigma^2 - \tilde{\sigma}^2\right) S^2 \Gamma^i\,dt + (\Delta^i - \Delta^a)\left((\mu - r + D)S\,dt + \sigma S\,dX\right).$$

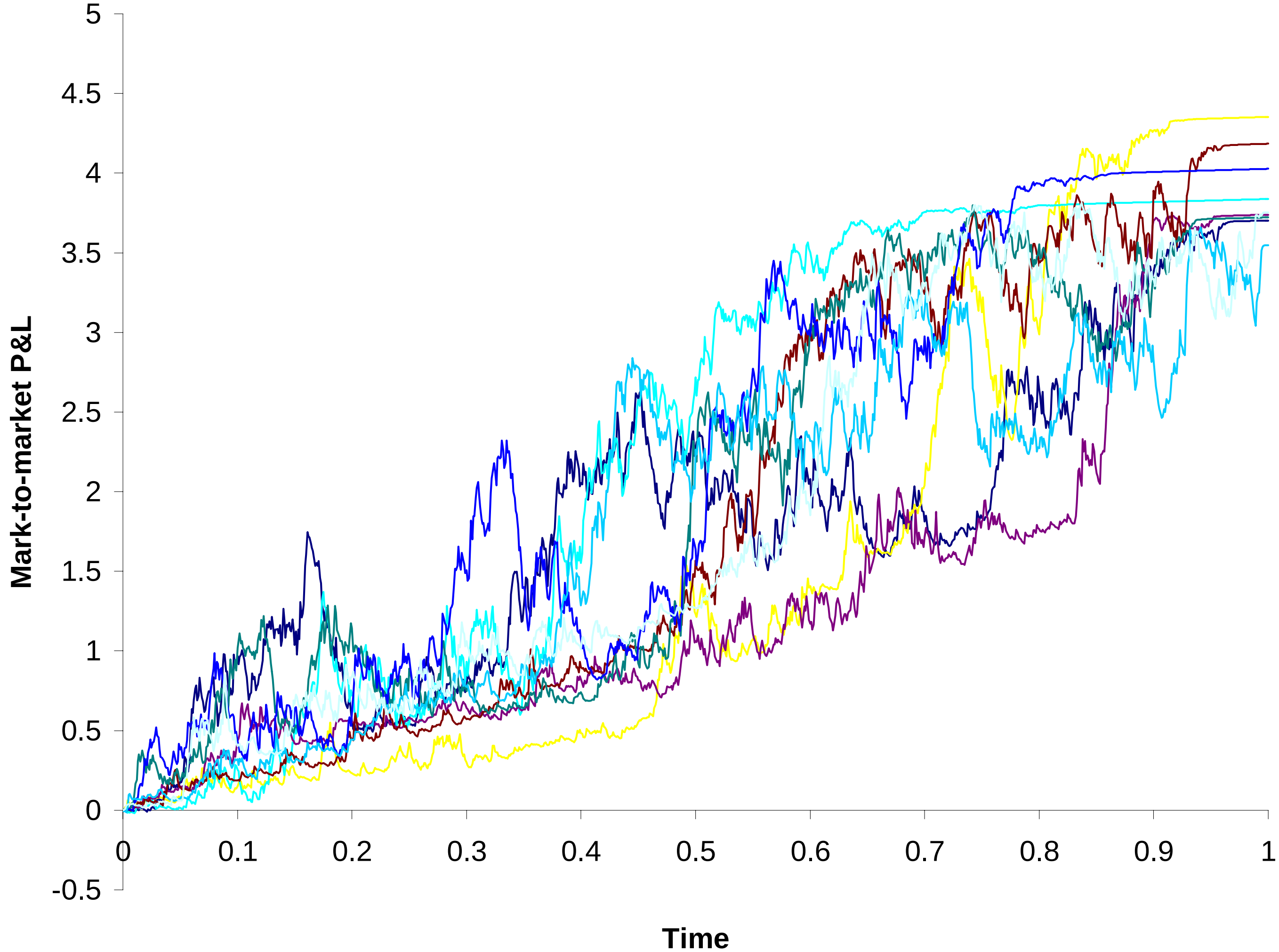
• The conclusion is that the final profit is guaranteed (the difference between the theoretical option values with the two volatilities) but how that is achieved is random, because of the $dX$ term in the above.

On a mark-to-market basis you could lose before you gain.

• Moreover, the mark-to-market profit depends on the real drift of the stock, $\mu$.

This is illustrated in the next figure. The figure shows several simulations of the same delta-hedged position.

Note that the final P&L is not *exactly* the same in each case because of the effect of hedging discretely, we hedged 'only' 1000 times for each realization. The option is a one-year European call, with a strike of 100, at the money initially, actual volatility is 30%, implied is 20%, the growth rate is 10% and interest rate 5%.




P&L for a delta-hedged option on a mark-to-market basis, hedged using actual volatility.

When $S$ changes, so will $V$. But these changes do not cancel each other out. From a risk management point of view this is not ideal.

• There is a simple analogy for this behavior. It is similar to owning a bond. For a bond there is a guaranteed outcome, but we may lose on a mark-to-market basis in the meantime.

## Case 2: Hedge with implied volatility, $\tilde{\sigma}$

Compare and contrast now with the case of hedging using a delta based on implied volatility.

• By hedging with implied volatility we are balancing the random fluctuations in the mark-to-market option value with the fluctuations in the stock price.

The evolution of the portfolio value is then 'deterministic' as we shall see.

Buy the option today, hedge using the implied delta, and put any cash in the bank earning $r$. The mark-to-market profit from today to tomorrow is

$$dV^i - \Delta^i \, dS - r(V^i - \Delta^i S) \, dt - \Delta^a DS \, dt$$

$$= \Theta^i \, dt + \tfrac{1}{2}\sigma^2 S^2 \Gamma^i \, dt - r(V^i - \Delta^i S) \, dt - \Delta^a DS \, dt$$

$$= \tfrac{1}{2}\left(\sigma^2 - \tilde{\sigma}^2\right) S^2 \Gamma^i \, dt. \qquad (1)$$

This is a far nicer way to make money.

• Observe how the profit is deterministic, there aren't any $dX$ terms.

From a risk management perspective this is much better behaved.

There is another advantage of hedging using implied volatility, we don't even need to know what actual volatility is.

- To make a profit all we need to know is that actual is always going to be greater than implied (if we are buying) or always less (if we are selling).

This takes some of the pressure off forecasting volatility accurately in the first place.

Add up the present value of all of these profits to get a total profit of

$$\frac{1}{2}\left(\sigma^2 - \tilde{\sigma}^2\right)\int_{t_0}^{T} e^{-r(t-t_0)} S^2 \Gamma^i \, dt.$$

- This is always positive, but highly path dependent.

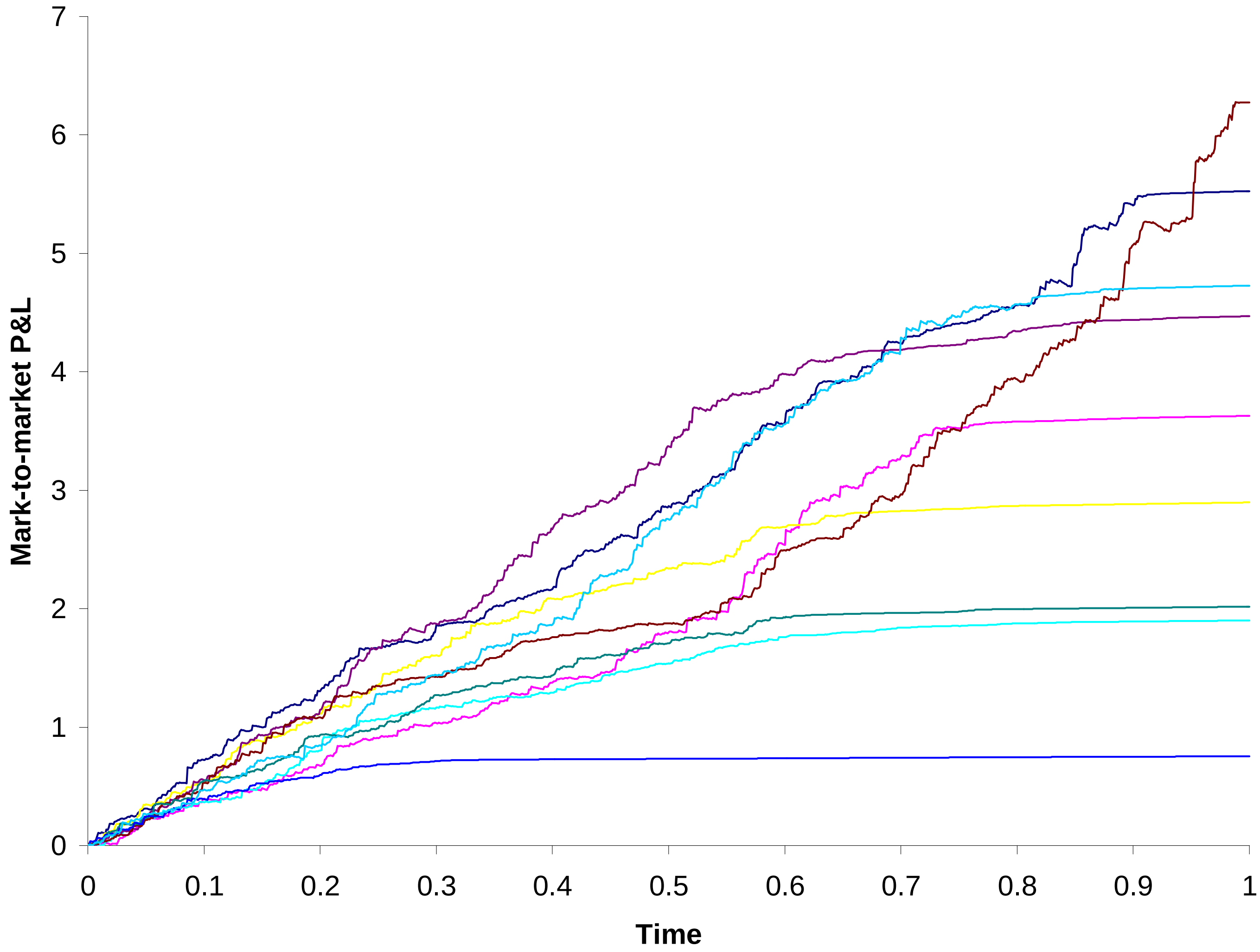
Being path dependent it will depend on the drift $\mu$.

If we start off at the money and the drift is very large (positive or negative) we will find ourselves quickly moving into territory where gamma and hence expression (1) is small, so that there will be not much profit to be made.

The best that could happen would be for the stock to end up close to the strike at expiration, this would maximize the total profit.

This path dependency is shown in the next figure.

The figure shows several realizations of the same delta-hedged position. Note that the lines are not perfectly smooth, again because of the effect of hedging discretely. The option and parameters are the same as in the previous example.




P&L for a delta-hedged option on a mark-to-market basis, hedged using implied volatility.

The simple analogy is now just putting money in the bank. The P&L is always increasing in value but the end result is random.

Carr (2005) and Henrard (2001) show the more general result that if you hedge using a delta based on a volatility $\sigma_h$ then the PV of the total profit is given by

$$V(S,t;\sigma_h) - V(S,t;\tilde{\sigma}) + \frac{1}{2}\left(\sigma^2 - \sigma_h^2\right)\int_{t_0}^{T} e^{-r(t-t_0)} S^2 \Gamma^h \, dt,$$

where the superscript on the gamma means that it uses the Black–Scholes formula with a volatility of $\sigma_h$.

## The expected profit after hedging using implied volatility

When you hedge using delta based on implied volatility the profit each 'day' is deterministic but the present value of total profit by expiration is path dependent, and given by

$$\frac{1}{2}\left(\sigma^2-\tilde{\sigma}^2\right)\int_{t_0}^{T} e^{-r(s-t_0)} S^2 \Gamma^i \, ds.$$

Introduce

$$I = \frac{1}{2}\left(\sigma^2 - \tilde{\sigma}^2\right)\int_{t_0}^{t} e^{-r(s-t_0)}S^2\Gamma^i\,ds.$$

Since therefore

$$dI = \frac{1}{2}\left(\sigma^2 - \tilde{\sigma}^2\right)\ e^{-r(t-t_0)}S^2\Gamma^i\,dt$$

we can write down the following partial differential equation for the *real* expected value, $P(S, I, t)$, of $I$

$$\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + \mu S \frac{\partial P}{\partial S} + \frac{1}{2}\left(\sigma^2 - \tilde{\sigma}^2\right)\ e^{-r(t-t_0)}S^2\Gamma^i\,\frac{\partial P}{\partial I} = 0,$$

with

$$P(S, I, T) = I.$$

Look for a solution of this equation of the form

$$P(S, I, t) = I + F(S, t)$$

so that

$$\frac{\partial F}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + \mu S \frac{\partial F}{\partial S} + \tfrac{1}{2}\left(\sigma^2 - \tilde{\sigma}^2\right)\ e^{-r(t-t_0)} S^2 \Gamma^i = 0.$$

The source term can be simplified to

$$\frac{E\left(\sigma^2 - \tilde{\sigma}^2\right) e^{-r(T-t_0)} e^{-d_2^2/2}}{2\tilde{\sigma}\sqrt{2\pi(T-t)}},$$

where

$$d_2 = \frac{\ln(S/E) + (r - D - \frac{1}{2}\tilde{\sigma}^2)(T-t)}{\sigma\sqrt{T-t}}.$$

Change variables to

$$x = \ln(S/E) + \frac{2}{\sigma^2}\left(\mu - \tfrac{1}{2}\sigma^2\right)\tau \quad \text{and} \quad \tau = \frac{\sigma^2}{2}(T - t),$$

where $E$ is the strike and $T$ is expiration, and write

$$F(S,t) = w(x,\tau).$$

The resulting partial differential equation is a then nicer.

**Result 1:** After some manipulations we end up with the expected profit initially ($t = t_0$, $S = S_0$, $I = 0$) being the single integral

$$F(S_0, t_0) = \frac{Ee^{-r(T-t_0)}(\sigma^2 - \tilde{\sigma}^2)}{2\sqrt{2\pi}} \int_{t_0}^{T} \frac{1}{\sqrt{\sigma^2(s-t_0) + \tilde{\sigma}^2(T-s)}}$$

$$\exp\left(-\frac{\left(\ln(S_0/E) + \left(\mu - \frac{1}{2}\sigma^2\right)(s-t_0) + \left(r - D - \frac{1}{2}\tilde{\sigma}^2\right)(T-s)\right)^2}{2(\sigma^2(s-t_0) + \tilde{\sigma}^2(T-s))}\right) ds.$$

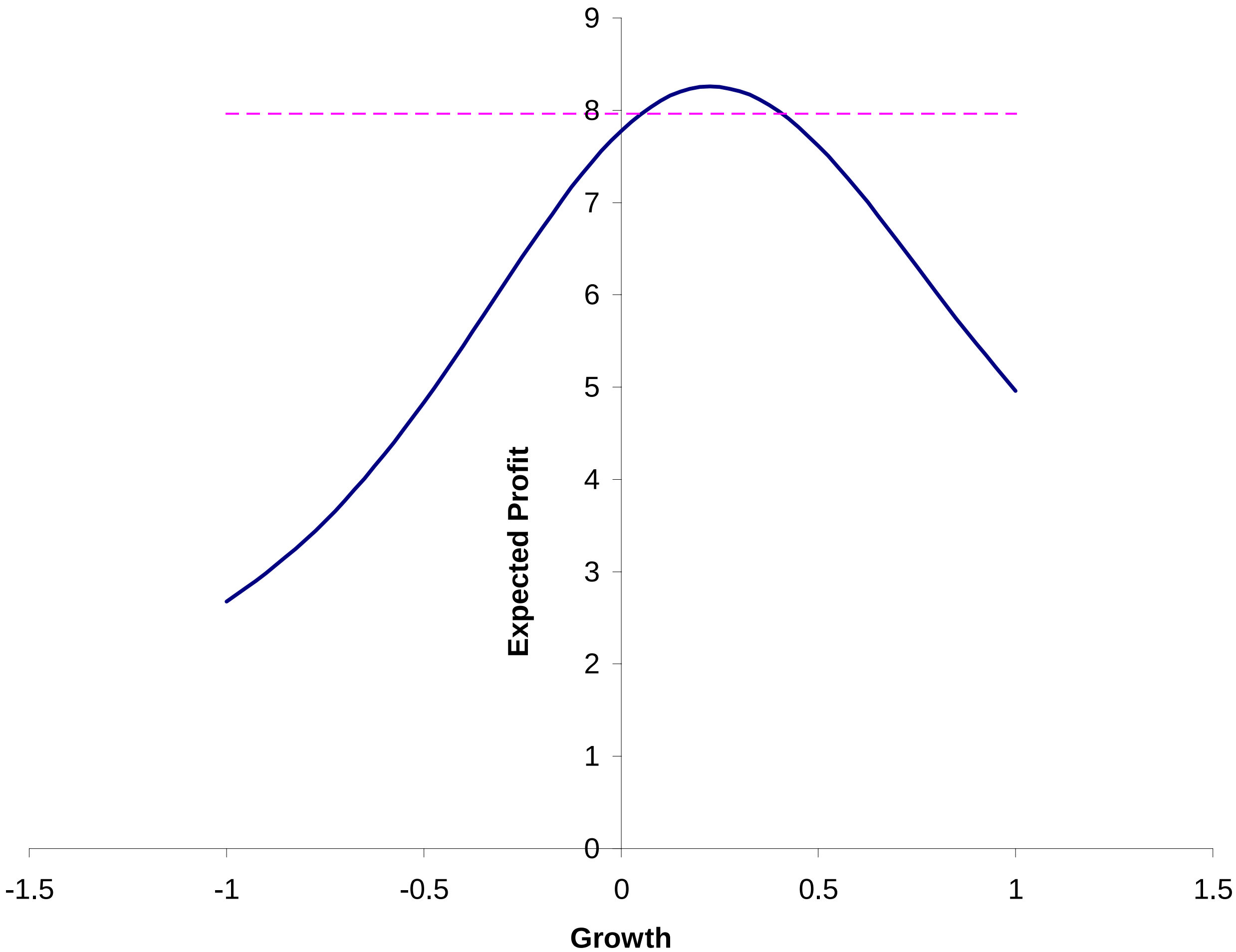
**Derivation:** See Appendix.

Results are shown in the following figures.

In the first figure is shown the expected profit versus the growth rate $\mu$. Parameters are $S = 100$, $\sigma = 0.4$, $r = 0.05$, $D = 0$, $E = 110$, $T = 1$, $\tilde{\sigma} = 0.2$.

- Observe that the expected profit has a maximum. This will be at the growth rate that ensures, roughly speaking, that the stock ends up close to at the money at expiration, where gamma is largest.

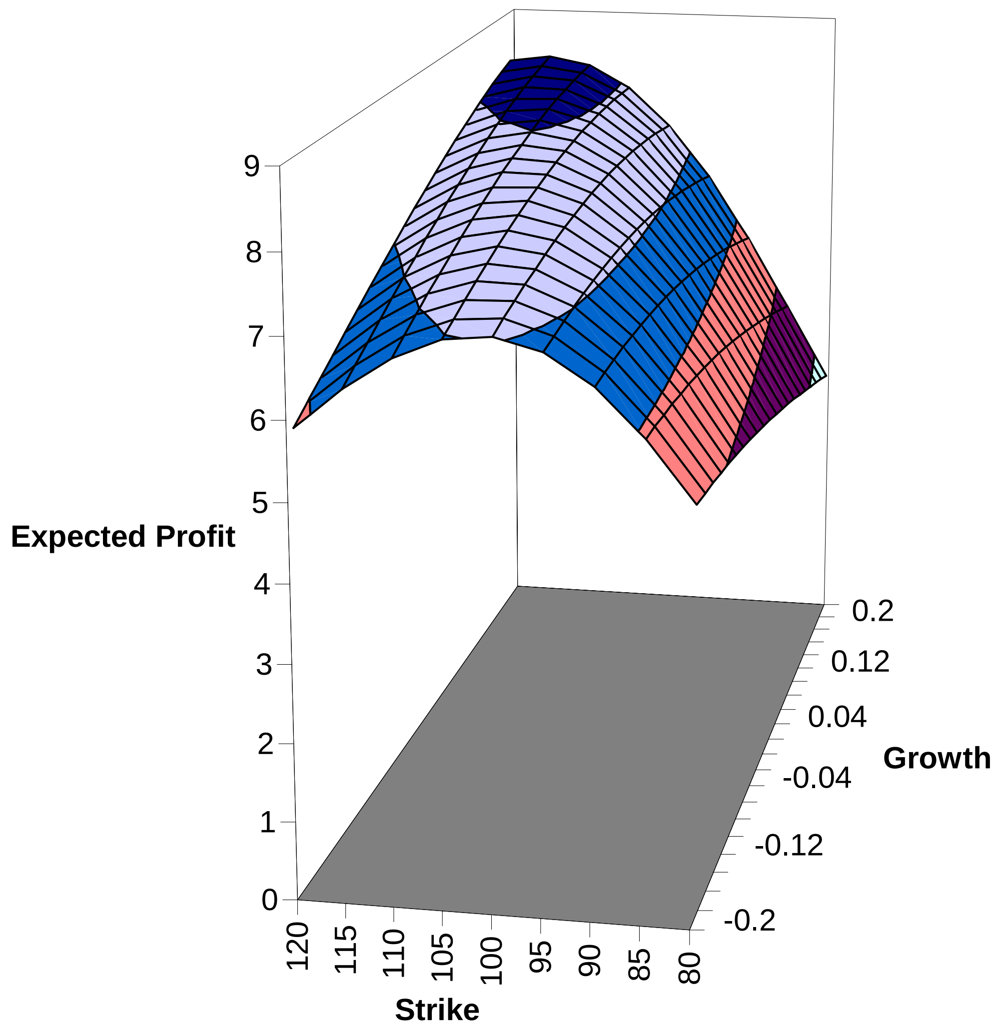
In the figure is also shown the profit to be made when hedging with actual volatility. For most realistic parameter regimes the maximum expected profit hedging with implied is similar to the guaranteed profit hedging with actual.



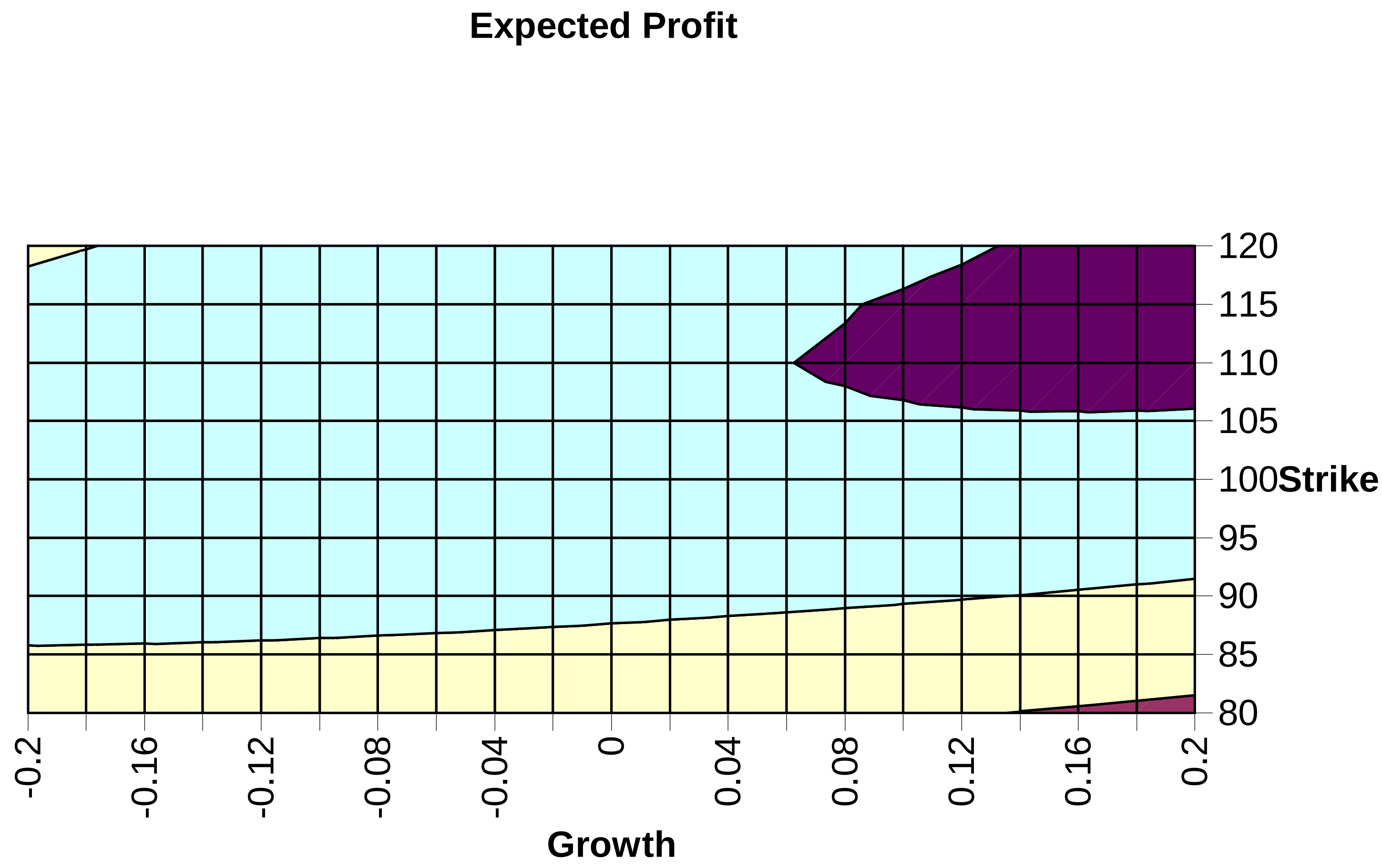

Expected profit, hedging using implied volatility, versus growth rate $\mu$; $S = 100$, $\sigma = 0.4$, $r = 0.05$, $D = 0$, $E = 110$, $T = 1$, $\tilde{\sigma} = 0.2$. The dashed line is the profit to be made when hedging with actual volatility.

In the next figure is shown expected profit versus $E$ and $\mu$. You can see how the higher the growth rate the larger the strike price at the maximum. The contour map is then shown.




Expected profit, hedging using implied volatility, versus growth rate $\mu$ and strike $E$; $S = 100$, $\sigma = 0.4$, $r = 0.05$, $D = 0$, $T = 1$, $\tilde{\sigma} = 0.2$.

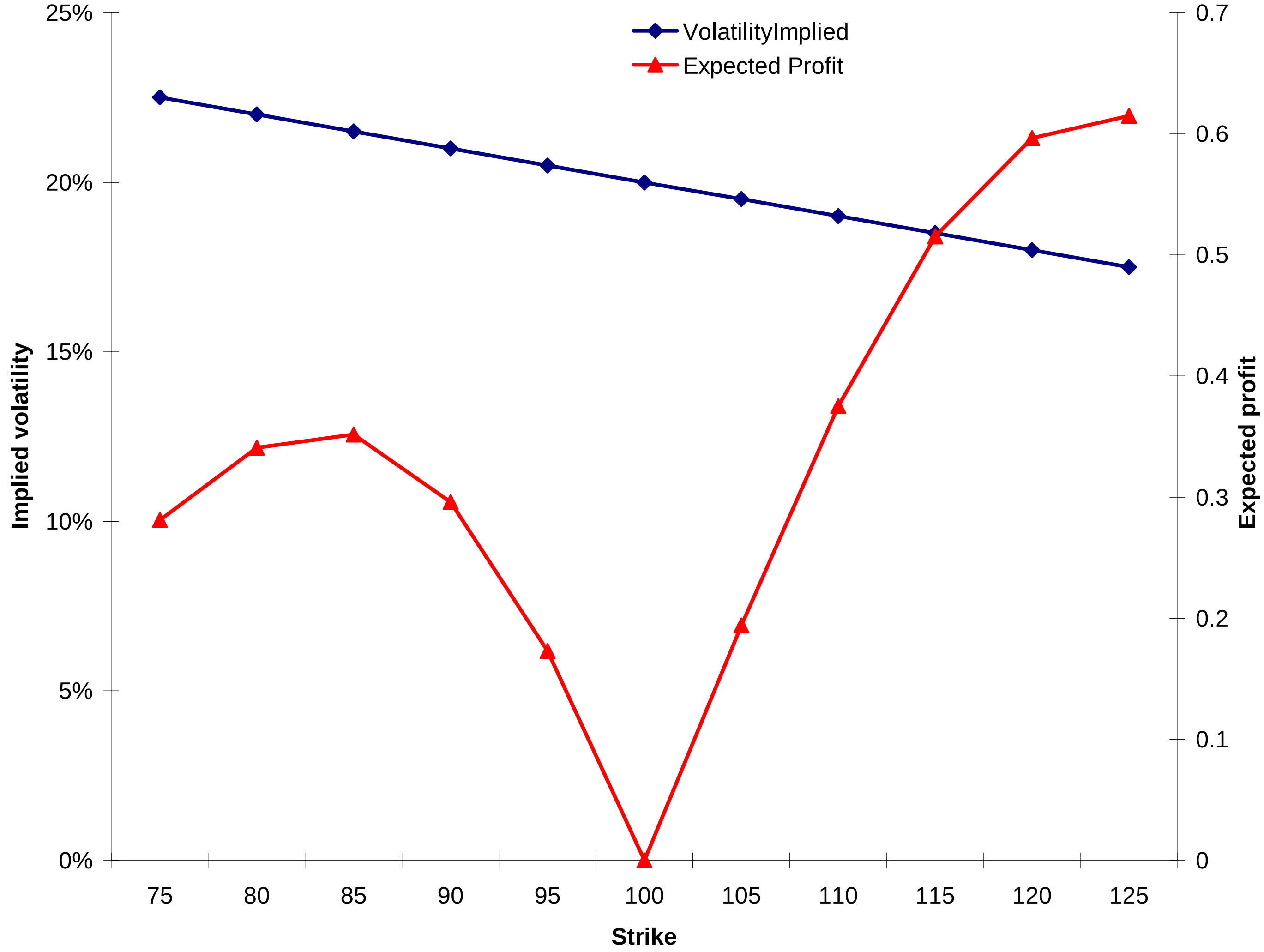



Contour map of expected profit, hedging using implied volatility, versus growth rate $\mu$ and strike $E$; $S = 100$, $\sigma = 0.4$, $r = 0.05$, $D = 0$, $T = 1$, $\tilde{\sigma} = 0.2$.

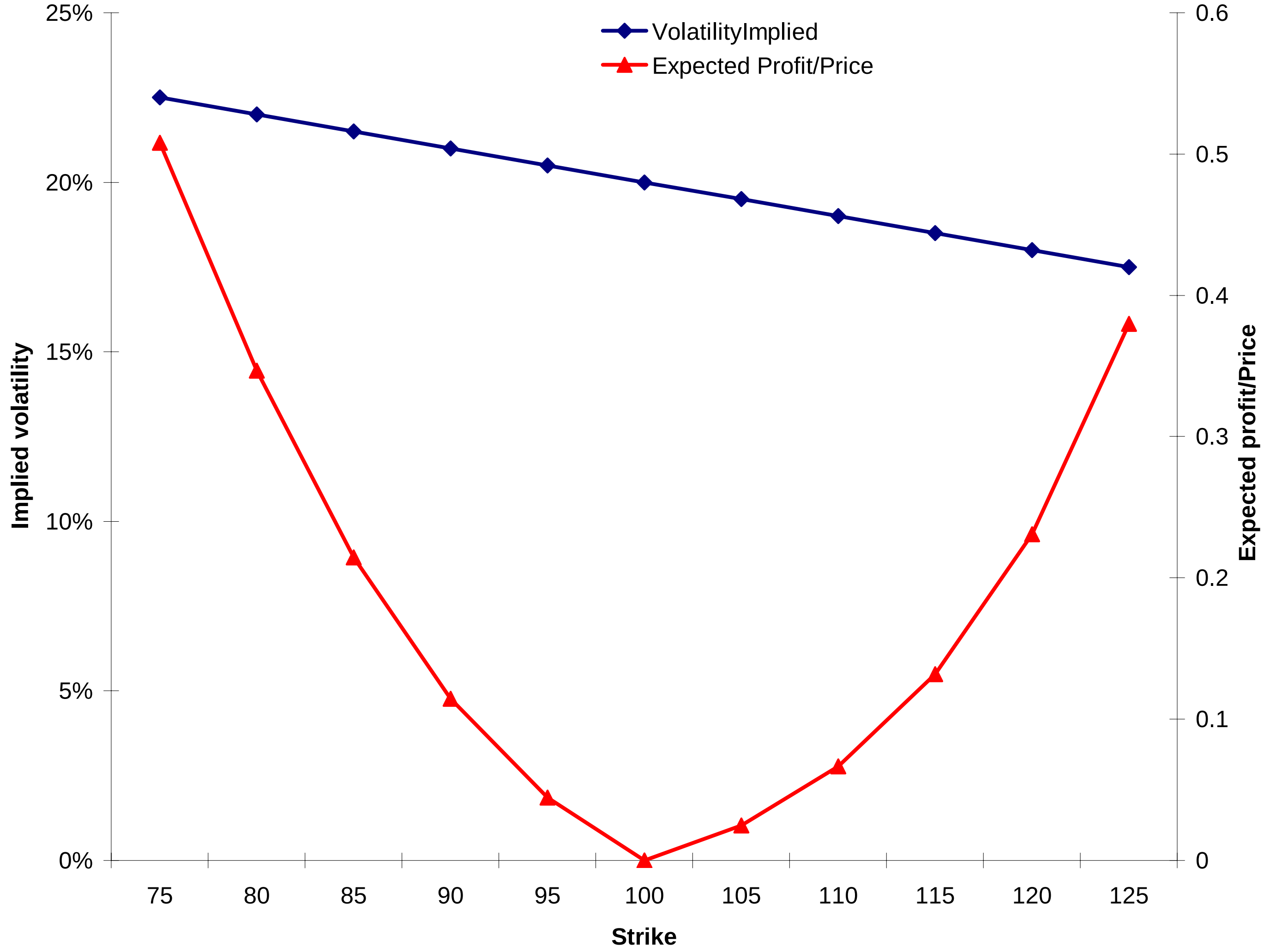
The effect of skew is shown next.

Here we have used a linear negative skew, from 22.5% at a strike of 75, falling to 17.5% at the 125 strike. The at-the-money implied volatility is 20% which in this case is the actual volatility.

This picture changes when you divide the expected profit by the price of the option (puts for lower strikes, call for higher). There is then no maximum, profitability increases with distance away from the money. Of course, this doesn't take into account the risk, the standard deviation associated with such trades.




Effect of skew, expected profit, hedging using implied volatility, versus strike $E$; $S = 100$, $\mu = 0$, $\sigma = 0.2$, $r = 0.05$, $D = 0$, $T = 1$.




Effect of skew, ratio of expected profit to price, hedging using implied volatility, versus strike $E$; $S = 100$, $\mu = 0$, $\sigma = 0.2$, $r = 0.05$, $D = 0$, $T = 1$.

## The variance of profit after hedging using implied volatility

Once we've calculated the expected profit from hedging using implied volatility we can calculate the variance in the final profit.

Using the above notation, the variance will be the expected value of $I^2$ less the square of the average of $I$, already calculated.

So we'll need to calculate $v(S,I,t)$ where

$$\frac{\partial v}{\partial t}+\tfrac{1}{2}\sigma^2S^2\frac{\partial^2 v}{\partial S^2}+\mu S\frac{\partial v}{\partial S}+\tfrac{1}{2}\left(\sigma^2-\tilde{\sigma}^2\right)\ e^{-r(t-t_0)}S^2\Gamma^i\ \frac{\partial v}{\partial I}=0,$$

with

$$v(S,I,T)=I^2.$$

The details of finding this function $v$ are rather messy, but a solution can be found of the form

$$v(S,I,t)=I^2+2I\ H(S,t)+G(S,t).$$

**Result 2:** The initial variance is $G(S_0, t_0) - F(S_0, t_0)^2$, where

$$G(S_0, t_0) = \frac{E^2(\sigma^2 - \tilde{\sigma}^2)^2 e^{-2r(T-t_0)}}{4\pi\sigma\tilde{\sigma}} \int_{t_0}^{T} \int_{s}^{T}$$

$$\frac{e^{p(u,s;S_0,t_0)}}{\sqrt{s-t_0}\sqrt{T-s}\sqrt{\sigma^2(u-s)+\tilde{\sigma}^2(T-u)}\sqrt{\frac{1}{\sigma^2(s-t_0)} + \frac{1}{\tilde{\sigma}^2(T-s)} + \frac{1}{\sigma^2(u-s)+\tilde{\sigma}^2(T-u)}}}\, du\, ds \tag{2}$$

where

$$p(u, s; S_0, t_0) = -\frac{1}{2}\frac{(x + \alpha(T-s))^2}{\tilde{\sigma}^2(T-s)} - \frac{1}{2}\frac{(x+\alpha(T-u))^2}{\sigma^2(u-s)+\tilde{\sigma}^2(T-u)}$$

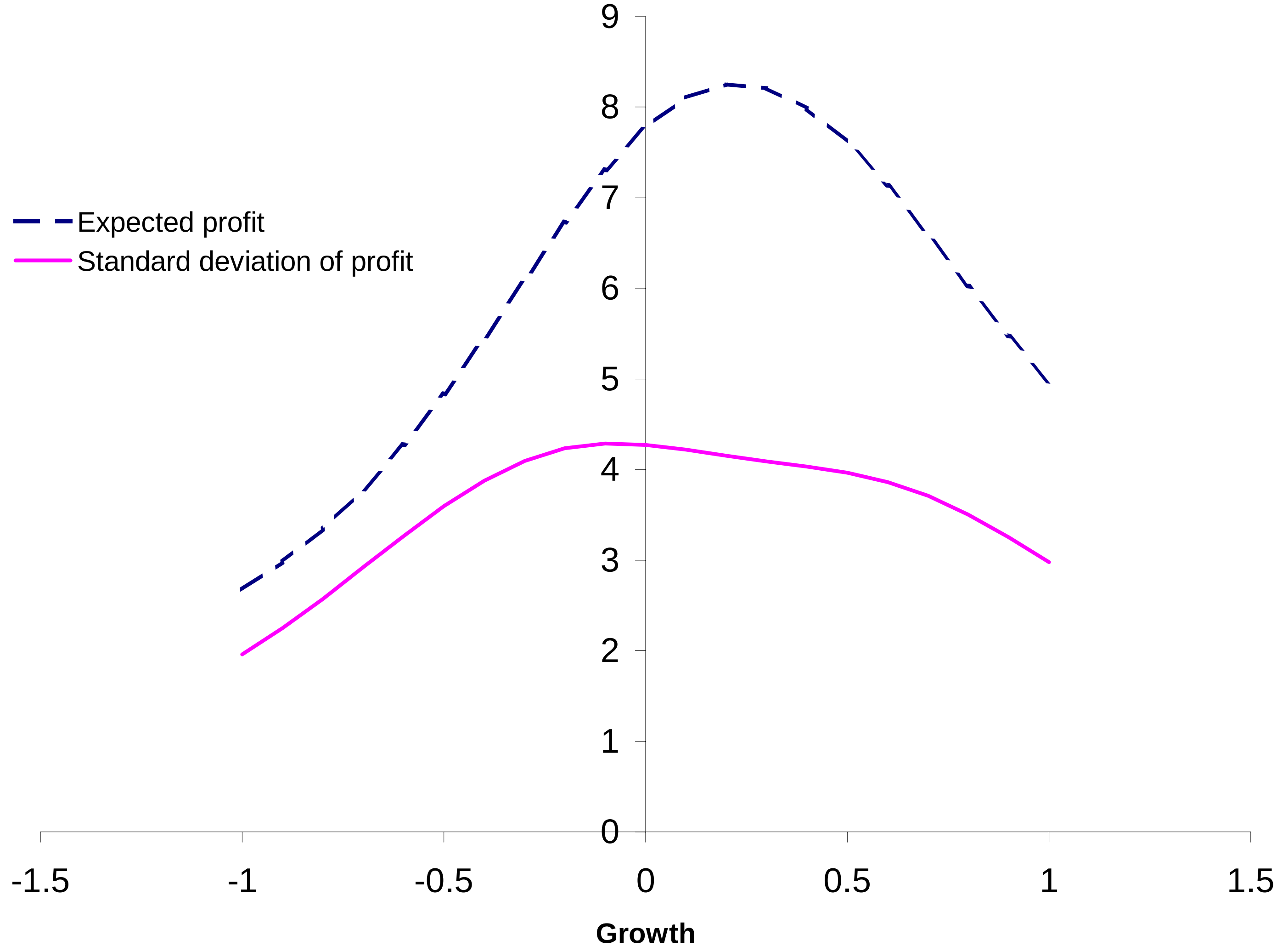
$$+\frac{1}{2}\frac{\left(\frac{x+\alpha(T-s)}{\tilde{\sigma}^2(T-s)} + \frac{x+\alpha(T-u)}{\sigma^2(u-s)+\tilde{\sigma}^2(T-u)}\right)^2}{\frac{1}{\sigma^2(s-t_0)} + \frac{1}{\tilde{\sigma}^2(T-s)} + \frac{1}{\sigma^2(u-s)+\tilde{\sigma}^2(T-u)}}$$

and

$$x = \ln(S_0/E) + \left(\mu - \tfrac{1}{2}\sigma^2\right)(T - t_0), \quad \text{and} \quad \alpha = \mu - \tfrac{1}{2}\sigma^2 - r + D + \tfrac{1}{2}\tilde{\sigma}^2.$$

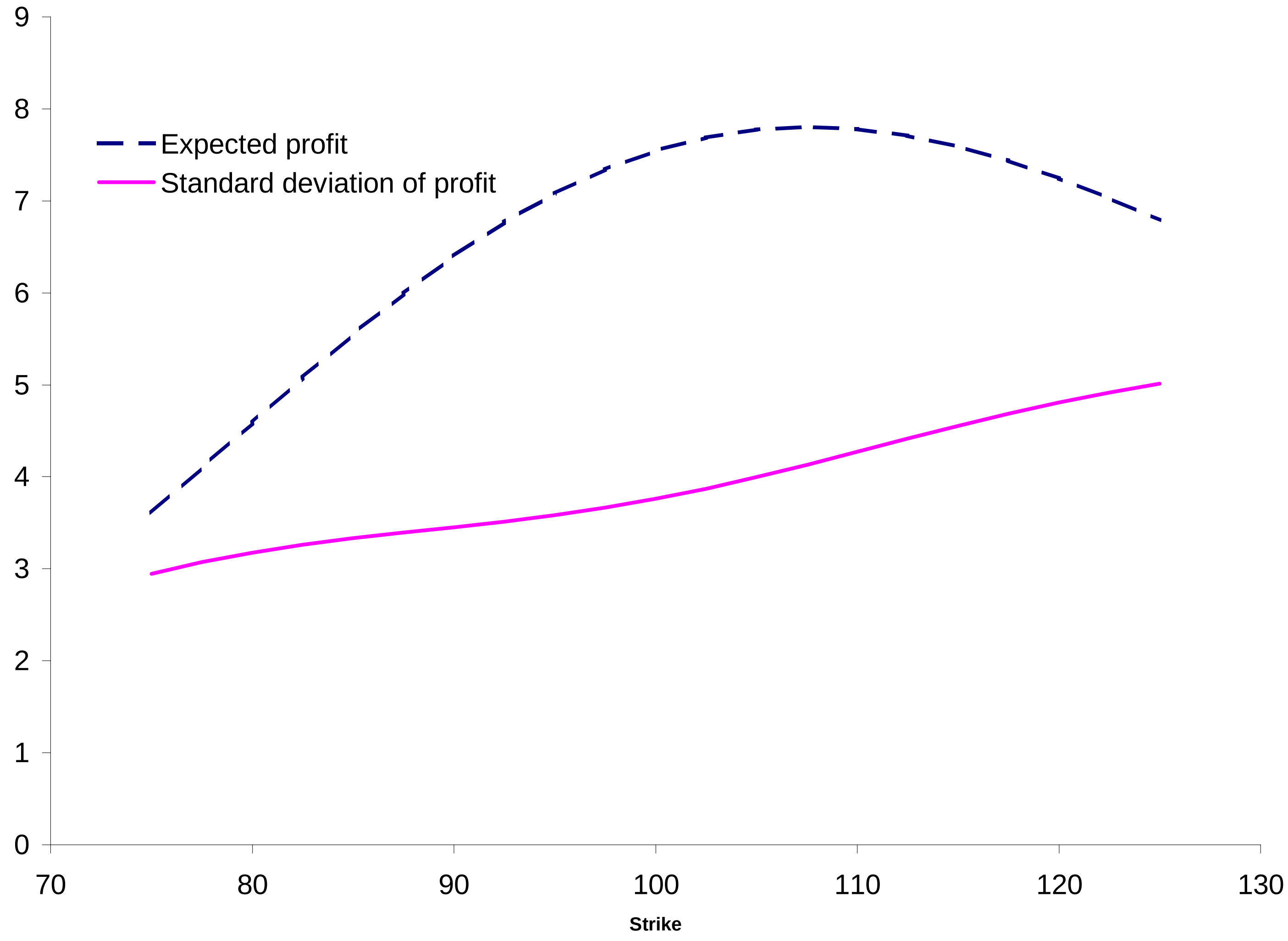
**Derivation:** See Appendix.

In the next figure is shown the standard deviation of profit versus growth rate, $S = 100$, $\sigma = 0.4$, $r = 0.05$, $D = 0$, $E = 110$, $T = 1$, $\tilde{\sigma} = 0.2$. And the following figure shows the standard deviation of profit versus strike, $S = 100$, $\sigma = 0.4$, $r = 0.05$, $D = 0$, $\mu = 0.1$, $T = 1$, $\tilde{\sigma} = 0.2$.




Standard deviation of profit, hedging using implied volatility, versus growth rate $\mu$; $S = 100$, $\sigma = 0.4$, $r = 0.05$, $D = 0$, $E = 110$, $T = 1$, $\tilde{\sigma} = 0.2$. (The expected profit is also shown.)

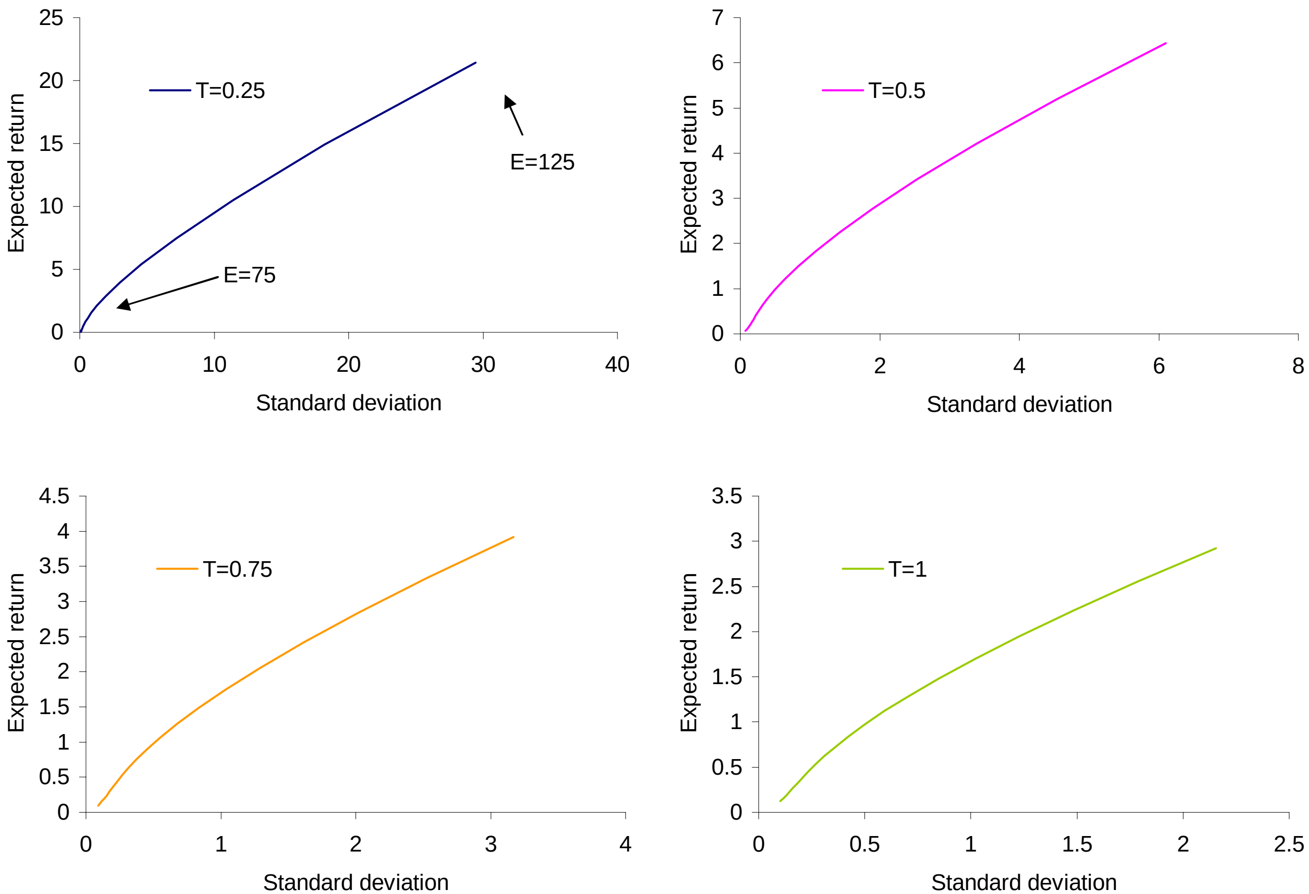



Standard deviation of profit, hedging using implied volatility, versus strike $E$; $S = 100$, $\sigma = 0.4$, $r = 0.05$, $D = 0$, $\mu = 0$, $T = 1$, $\tilde{\sigma} = 0.2$. (The expected profit is also shown.)

Note that in these plots the expectations and standard deviations have not been scaled with the cost of the options.

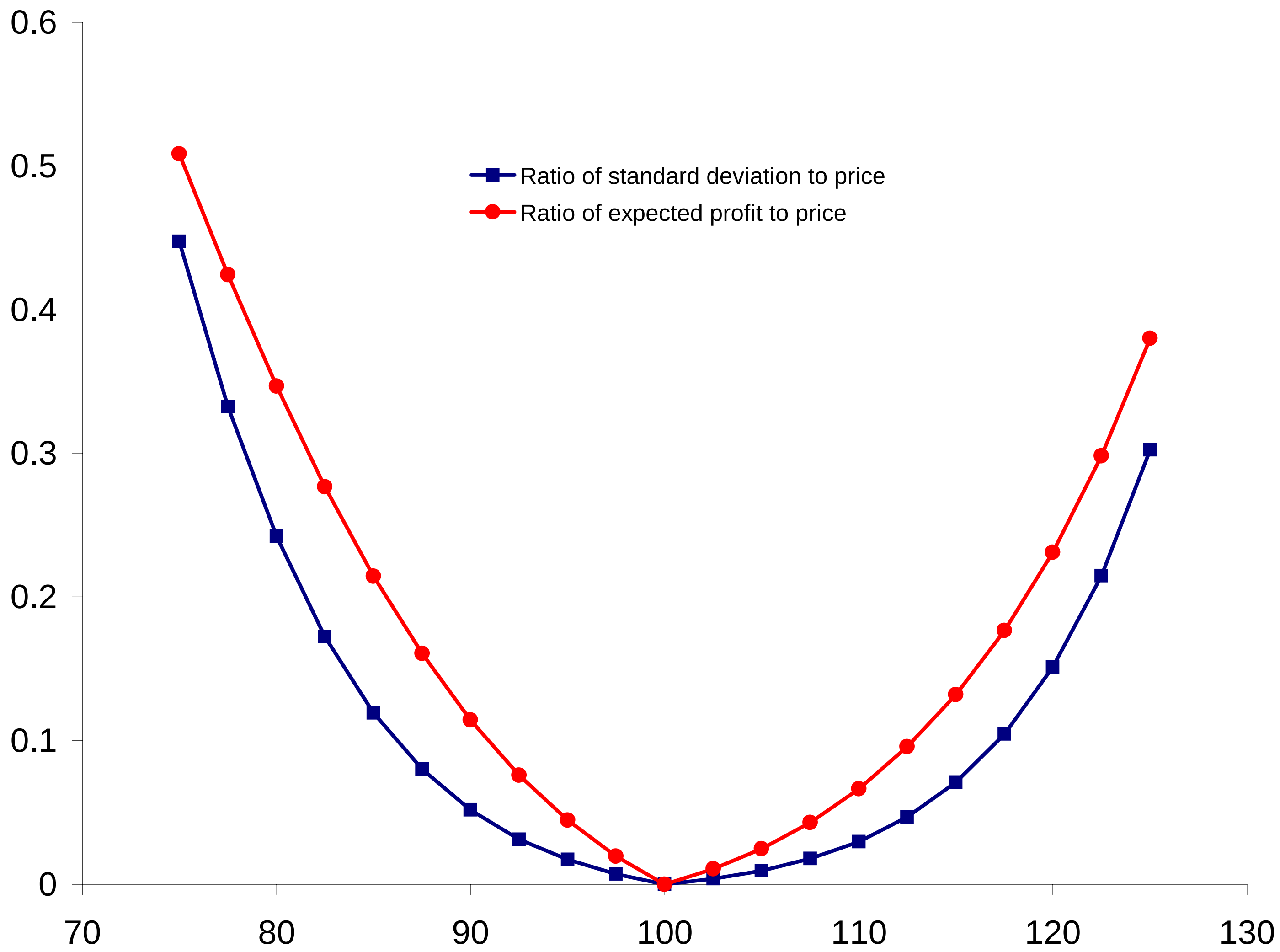
In the next figure is shown expected profit divided by cost versus standard deviation divided by cost, as both strike and expiration vary. In these $S = 100$, $\sigma = 0.4$, $r = 0.05$, $D = 0$, $\mu = 0.1$, $\tilde{\sigma} = 0.2$.

- To some extent, although we emphasis only *some*, these diagrams can be interpreted in a classical mean-variance manner. The main criticism is, of course, that we are not working with normal distributions, and, furthermore, there is no downside, no possibility of any losses.




Scaled expected profit versus scaled standard deviation; $S = 100$, $\sigma = 0.4$, $r = 0.05$, $D = 0$, $\mu = 0.1$, $\tilde{\sigma} = 0.2$. Four different expirations, varying strike.)

The next figure completes the earlier picture for the skew, since it now contains the standard deviation.




Effect of skew, ratio of expected profit to price, and ratio of standard deviation to price, versus strike $E$; $S = 100$, $\mu = 0$, $\sigma = 0.2$, $r = 0.05$, $D = 0$, $T = 1$.

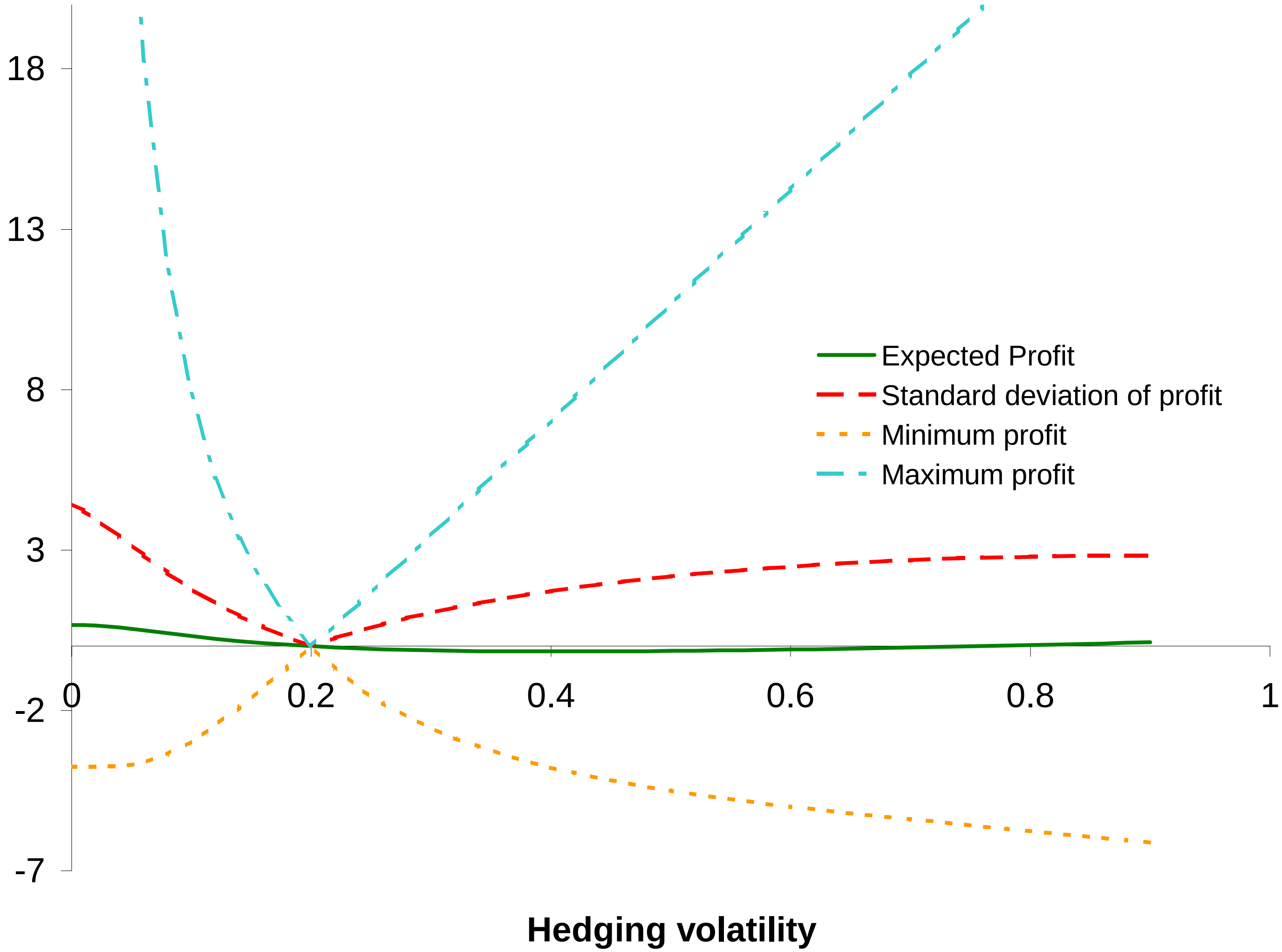
## Hedging with different volatilities

We will briefly examine hedging using volatilities other than actual or implied.

**Actual volatility = Implied volatility**

For the first example let's look at hedging a long position in a correctly priced option, so that $\sigma = \tilde{\sigma}$. We will hedge using different volatilities, $\sigma^h$. Results are shown in the next figure. The figure shows the expected profit and standard deviation of profit when hedging with various volatilities. The chart also shows minimum and maximum profit. Parameters are $E = 100$, $S = 100$, $\mu = 0$, $\sigma = 0.2$, $r = 0.1$, $D = 0$, $T = 1$, and $\tilde{\sigma} = 0.2$.

With these parameters the expected profit is small as a fraction of the market price of the option ($13.3) regardless of the hedging volatility. The standard deviation of profit is zero when the option is hedged at the actual volatility. The upside, the maximum profit is much greater than the downside. Crucially all of the curves have zero value at the actual/implied volatility.

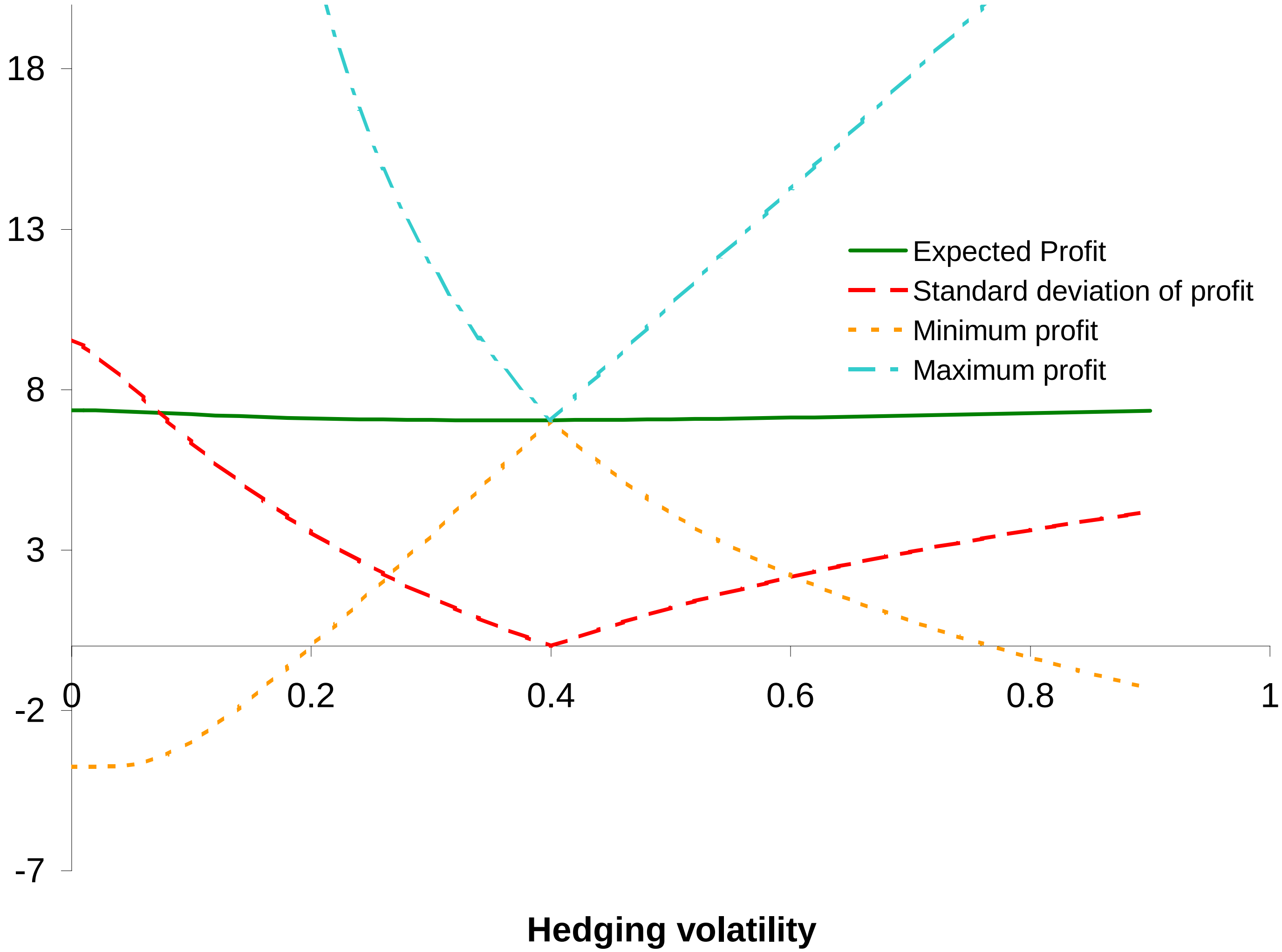



Expected profit, standard deviation of profit, minimum and maximum, hedging with various volatilities. $E = 100$, $S = 100$, $\mu = 0$, $\sigma = 0.2$, $r = 0.1$, $D = 0$, $T = 1$, $\tilde{\sigma} = 0.2$.

**Actual volatility $>$ Implied volatility**

In the next figure is shown the expected profit and standard deviation of profit when hedging with various volatilities when actual volatility is greater than implied. The chart again also shows minimum and maximum profit. Parameters are $E = 100$, $S = 100$, $\mu = 0$, $\sigma = 0.4$, $r = 0.1$, $D = 0$, $T = 1$, and $\tilde{\sigma} = 0.2$.

Note that it is possible to lose money if you hedge at below implied, but hedging with a higher volatility you will not be able to lose until hedging with a volatility of approximately 75%. The expected profit is again insensitive to hedging volatility.

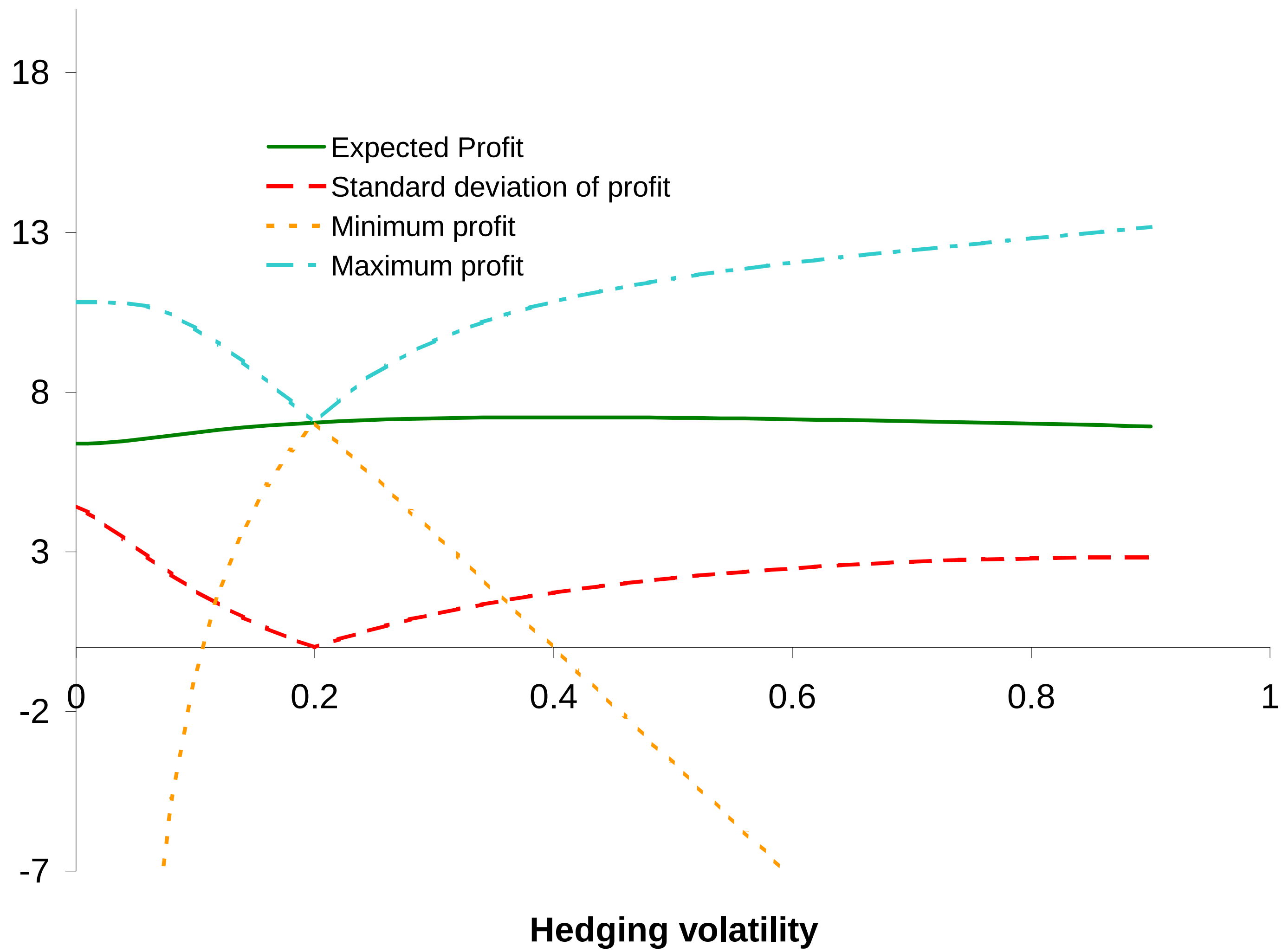



Expected profit, standard deviation of profit, minimum and maximum, hedging with various volatilities. $E = 100$, $S = 100$, $\mu = 0$, $\sigma = 0.4$, $r = 0.1$, $D = 0$, $T = 1$, $\tilde{\sigma} = 0.2$.

## Actual volatility $<$ Implied volatility

In the next figure is shown properties of the profit when hedging with various volatilities when actual volatility is less than implied. We are now selling the option and delta hedging it. Parameters are $E = 100$, $S = 100$, $\mu = 0$, $\sigma = 0.4$, $r = 0.1$, $D = 0$, $T = 1$, and $\tilde{\sigma} = 0.2$.

Now it is possible to lose money if you hedge at above implied, but hedging with a lower volatility you will not be able to lose until hedging with a volatility of approximately 10%. The expected profit is again insensitive to hedging volatility. The downside is now more dramatic than the upside.




Expected profit, standard deviation of profit, minimum and maximum, hedging with various volatilities. $E = 100$, $S = 100$, $\mu = 0$, $\sigma = 0.2$, $r = 0.1$, $D = 0$, $T = 1$, $\tilde{\sigma} = 0.4$.

## Pros and cons of hedging with each volatility

Given that we seem to have a choice in how to delta hedge it is instructive to summarize the advantages and disadvantages of the possibilities.

## Hedging with actual volatility

**Pros:**

- The main advantage of hedging with actual volatility is that you know exactly what profit you will get at expiration.

**Cons:**

- The P&L fluctuations during the life of the option can be daunting, and so less appealing from a 'local' as opposed to 'global' risk management perspective.

- You are unlikely to be totally confident in your volatility forecast, the number you are putting into your delta formula.

## Hedging with implied volatility

**Pros:**

- There are no local fluctuations in P&L, you are continually making a profit.
- You only need to be on the right side of the trade to profit. Buy when actual is going to be higher than implied and sell if lower.
- The number that goes into the delta is implied volatility, and therefore easy to observe.

**Cons:**

- You don't know how much money you will make, only that it is positive.

## Hedging with another volatility

You can obviously balance the pros and cons of hedging with actual and implied by hedging with another volatility altogether. See Dupire (2005) for work in this area.

In practice which volatility one uses is often determined by whether one is constrained to mark to market or mark to model. If one is able to mark to model then one is not necessarily concerned with the day-to-day fluctuations in the mark-to-market profit and loss and so it is natural to hedge using actual volatility.

However, it is common to have to report profit and loss based on market values. This constraint may be imposed by a risk management department, by prime brokers, or by investors who may monitor the mark-to-market profit on a regular basis.

In this case it is more usual to hedge based on implied volatility to avoid the daily fluctuations in the profit and loss.

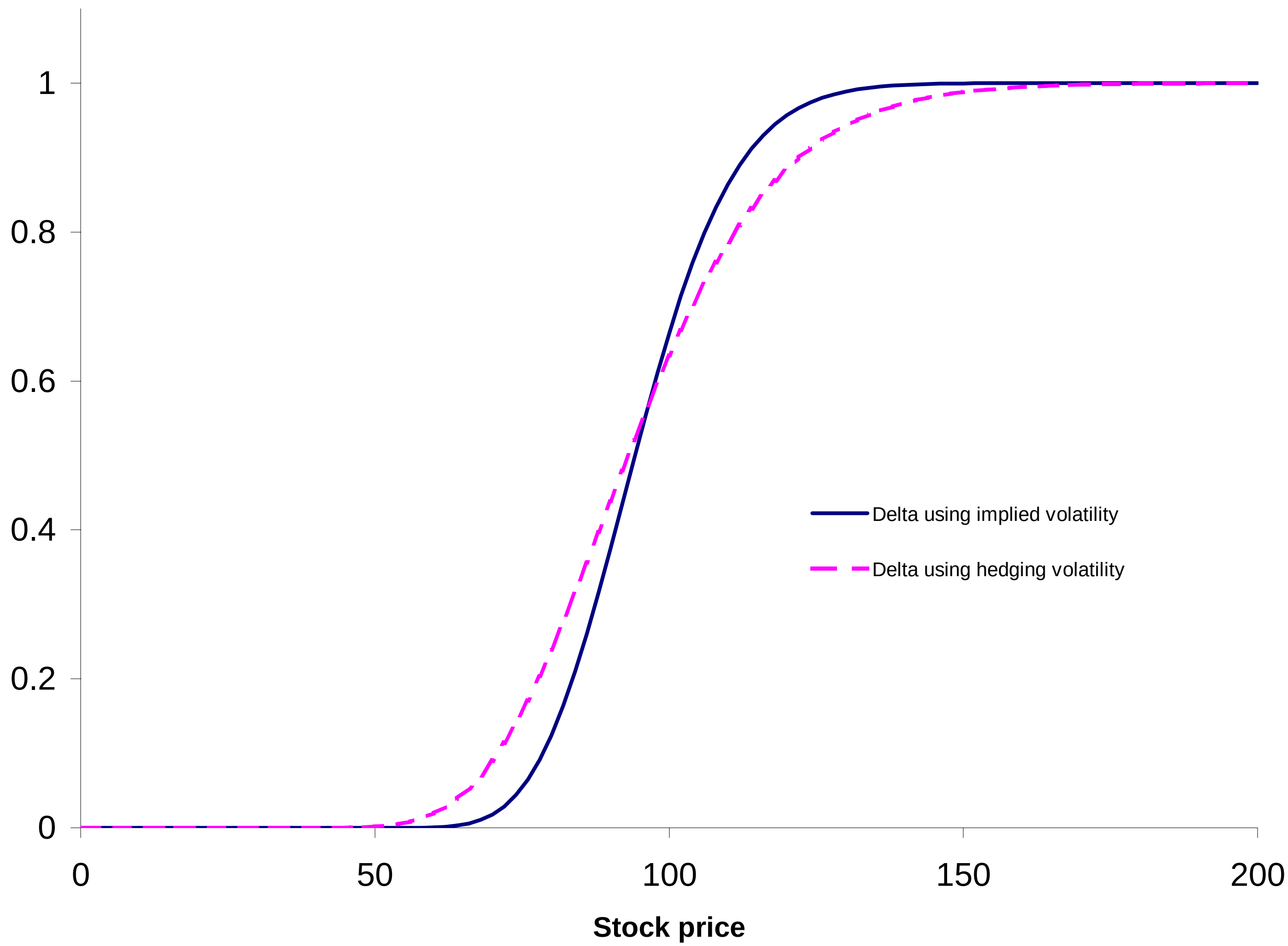
We can begin to quantify the 'local' risk, the daily fluctuations in P&L, by looking at the random component in a portfolio hedged using a volatility of $\sigma^h$.

The standard deviation of this risk is

$$\sigma S \mid \Delta^i - \Delta^h \mid \sqrt{dt}. \qquad (3)$$

Note that this expression depends on all three volatilities.

The next figure shows the two deltas (for a call option), one using implied volatility and the other the hedging volatility, six months before expiration.




Deltas based on implied volatility and hedging volatility. $E = 100$, $S = 100$, $r = 0.1$, $D = 0$, $T = 0.5$, $\tilde{\sigma} = 0.2$, $\sigma^h = 0.3$.

If the stock is far in or out of the money the two deltas are similar and so the local risk is small.

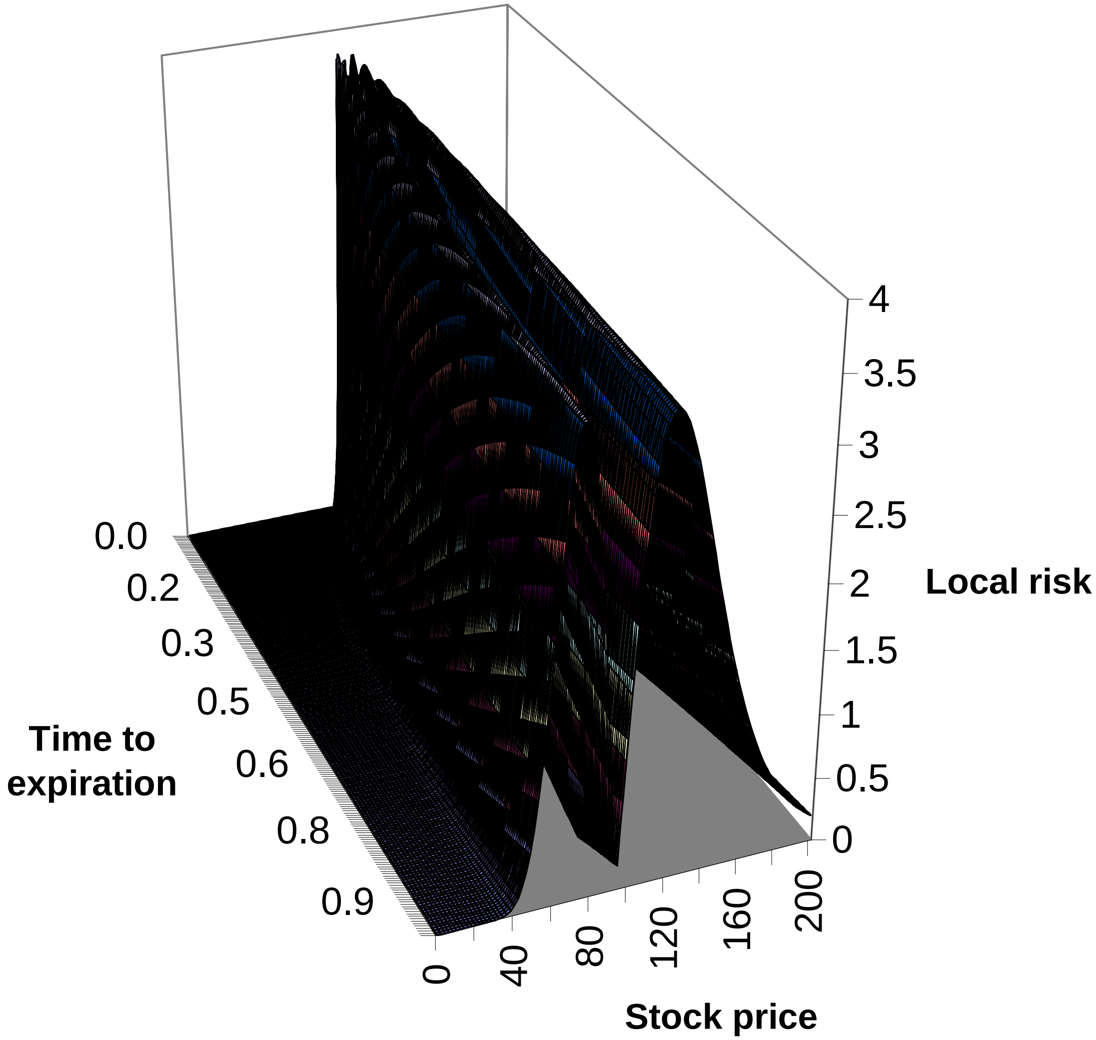
The local risk is also small where the two deltas cross over. This 'sweet spot' is at

$$\frac{\ln(S/E) + (r - D + \tilde{\sigma}^2/2)(T-t)}{\tilde{\sigma}\sqrt{T-t}} = \frac{\ln(S/E) + (r - D + {\sigma^h}^2/2)(T-t)}{\sigma^h\sqrt{T-t}},$$

that is,

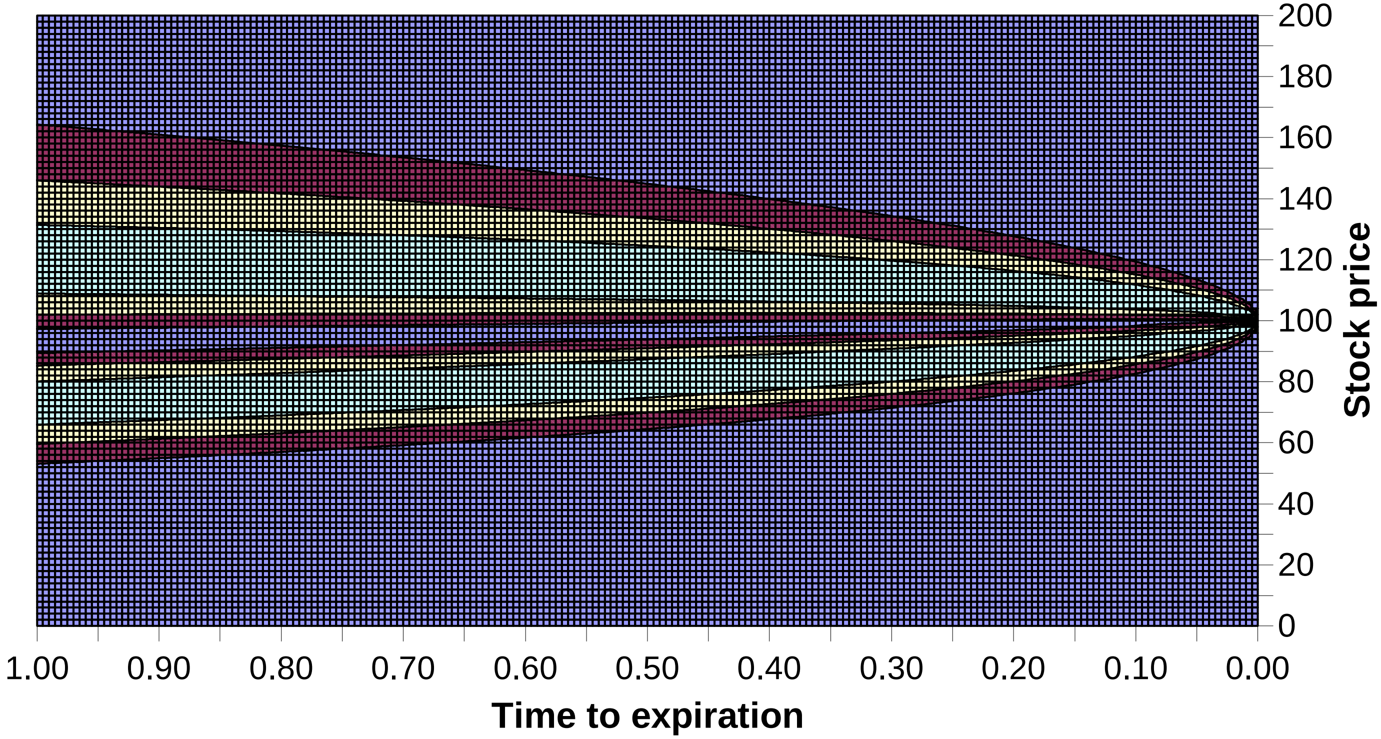
$$S = E \exp\left(-\frac{T-t}{\tilde{\sigma}-\sigma^h}\left(\tilde{\sigma}(r - D + {\sigma^h}^2/2) - \sigma^h(r - D + \tilde{\sigma}^2/2)\right)\right).$$

The next figure shows a three-dimensional plot of expression (3), without the $\sqrt{dt}$ factor, as a function of stock price and time. Following that is a contour map of the same. Parameters are $E = 100$, $S = 100$, $\sigma = 0.4$, $r = 0.1$, $D = 0$, $T = 1$, $\tilde{\sigma} = 0.2$, $\sigma^h = 0.3$.




Local risk as a function of stock price and time to expiration. $E = 100$, $S = 100$, $\sigma = 0.4$, $r = 0.1$, $D = 0$, $T = 1$, $\tilde{\sigma} = 0.2$, $\sigma^h = 0.3$.




Contour map of local risk as a function of stock price and time to expiration. $E = 100$, $S = 100$, $\sigma = 0.4$, $r = 0.1$, $D = 0$, $T = 1$, $\tilde{\sigma} = 0.2$, $\sigma^h = 0.3$.

For the remainder of this lecture we will only consider the case of hedging using a delta based on implied volatility.

## Portfolios when hedging with implied volatility

A natural extension to the above analysis is to look at portfolios of options, options with different strikes and expirations.

Since only an option's gamma matters when we are hedging using implied volatility, calls and puts are effectively the same since they have the same gamma.

The profit from a portfolio is now

$$\frac{1}{2}\sum_k q_k \left(\sigma^2 - \tilde{\sigma}_k^2\right) \int_{t_0}^{T_k} e^{-r(s-t_0)} S^2 \Gamma_k^i \, ds,$$

where $k$ is the index for an option, and $q_k$ is the quantity of that option.

Introduce

$$I = \frac{1}{2} \sum_k q_k \left( \sigma^2 - \tilde{\sigma}_k^2 \right) \int_{t_0}^{t} e^{-r(s-t_0)} S^2 \Gamma_k^i \, ds, \qquad (4)$$

as a new state variable, and the analysis can proceed as before. Note that since there may be more than one expiration date since we have several different options, it must be understood in Equation (4) that $\Gamma_k^i$ is zero for times beyond the expiration of the option.

The governing differential operator for expectation, variance, etc. is then

$$\frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + \mu S \frac{\partial}{\partial S} + \frac{1}{2}\sum_k \left(\sigma^2 - \tilde{\sigma}_k^2\right)\ e^{-r(t-t_0)} S^2 \Gamma_k^i \frac{\partial}{\partial I} = 0,$$

with final condition representing expectation, variance, etc.

## Expectation

**Result 3:** The solution for the present value of the expected profit ($t = t_0$, $S = S_0$, $I = 0$) is simply the sum of individual profits for each option,

$$F(S_0, t_0) = \sum_k q_k \frac{E_k e^{-r(T_k - t_0)}(\sigma^2 - \tilde{\sigma}_k^2)}{2\sqrt{2\pi}} \int_{t_0}^{T_k} \frac{1}{\sqrt{\sigma^2(s - t_0) + \tilde{\sigma}_k^2(T_k - s)}}$$

$$\exp\left(-\frac{\left(\ln(S_0/E_k) + \left(\mu - \frac{1}{2}\sigma^2\right)(s - t_0) + \left(r - D - \frac{1}{2}\tilde{\sigma}_k^2\right)(T_k - s)\right)^2}{2(\sigma^2(s - t_0) + \tilde{\sigma}_k^2(T_k - s))}\right) ds.$$

**Derivation:** See Appendix.

## Variance

**Result 4:** The variance is more complicated, obviously, because of the correlation between all of the options in the portfolio. Nevertheless, we can find an expression for the initial variance as $G(S_0, t_0) - F(S_0, t_0)^2$ where

$$G(S_0, t_0) = \sum_j \sum_k q_j q_k G_{jk}(S_0, t_0)$$

where

$$G_{jk}(S_0, t_0) = \frac{E_j E_k (\sigma^2 - \tilde{\sigma}_j^2)(\sigma^2 - \tilde{\sigma}_k^2) e^{-r(T_j - t_0) - r(T_k - t_0)}}{4\pi\sigma\tilde{\sigma}_k} \int_{t_0}^{\min(T_j, T_k)} \int_s^{T_j}$$

$$\frac{e^{p(u,s;S_0,t_0)}}{\sqrt{s-t_0}\sqrt{T_k - s}\sqrt{\sigma^2(u-s) + \tilde{\sigma}_j^2(T_j - u)}\sqrt{\frac{1}{\sigma^2(s-t_0)} + \frac{1}{\tilde{\sigma}_k^2(T_k - s)} + \frac{1}{\sigma^2(u-s) + \tilde{\sigma}_j^2(T_j - u)}}} \, du \, ds \tag{5}$$

where

$$p(u,s;S_0,t_0) = -\frac{1}{2}\frac{(\ln(S_0/E_k)+\bar{\mu}(s-t_0)+\bar{r}_k(T_k-s))^2}{\tilde{\sigma}_k^2(T_k-s)}$$

$$-\frac{1}{2}\frac{(\ln(S_0/E_j)+\bar{\mu}(u-t_0)+\bar{r}_j(T_j-u))^2}{\sigma^2(u-s)+\tilde{\sigma}_j^2(T_j-u)}$$

$$+\frac{1}{2}\frac{\left(\frac{\ln(S_0/E_k)+\bar{\mu}(s-t_0)+\bar{r}_k(T_k-s)}{\tilde{\sigma}_k^2(T_k-s)}+\frac{\ln(S_0/E_j)+\bar{\mu}(u-t_0)+\bar{r}_j(T_j-u)}{\sigma^2(u-s)+\tilde{\sigma}_j^2(T_j-u)}\right)^2}{\frac{1}{\sigma^2(s-t_0)}+\frac{1}{\tilde{\sigma}_k^2(T_k-s)}+\frac{1}{\sigma^2(u-s)+\tilde{\sigma}_j^2(T_j-u)}}$$

and

$$\bar{\mu} = \mu - \frac{1}{2}\sigma^2, \quad \bar{r}_j = r - D - \frac{1}{2}\tilde{\sigma}_j^2 \quad \text{and} \quad \bar{r}_k = r - D - \frac{1}{2}\tilde{\sigma}_k^2.$$

**Derivation:** See Appendix.

**Portfolio optimization possibilities**

There is clearly plenty of scope for using the above formulas in portfolio optimization problems. Here we give one example.

The stock is currently at 100. The growth rate is zero, actual volatility is 20%, zero dividend yield and the interest rate is 5%.

The next table shows the available options, and associated parameters. Observe the negative skew.

The out-of-the-money puts are overvalued and the out-of-the-money calls are undervalued. (The 'Profit Total Expected' row assumes that we buy a single one of that option.)

| | A | B | C | D | E |
|---|---|---|---|---|---|
| **Type** | Put | Put | Call | Call | Call |
| **Strike** | 80 | 90 | 100 | 110 | 120 |
| **Expiration** | 1 | 1 | 1 | 1 | 1 |
| **Volatility, Implied** | 0.250 | 0.225 | 0.200 | 0.175 | 0.150 |
| **Option Price, Market** | 1.511 | 3.012 | 10.451 | 5.054 | 1.660 |
| **Option Value, Theory** | 0.687 | 2.310 | 10.451 | 6.040 | 3.247 |
| **Profit Total Expected** | -0.933 | -0.752 | 0.000 | 0.936 | 1.410 |

Available options.

Using the above formulas we can find the portfolio that maximizes or minimizes target quantities (expected profit, standard deviation, ratio of profit to standard deviation).

• Let us consider the simple case of maximizing the expected return, while constraining the standard deviation to be one.

This is a very natural strategy when trying to make a profit from volatility arbitrage while meeting constraints imposed by regulators, brokers, investors etc.

The result is given in the next table.

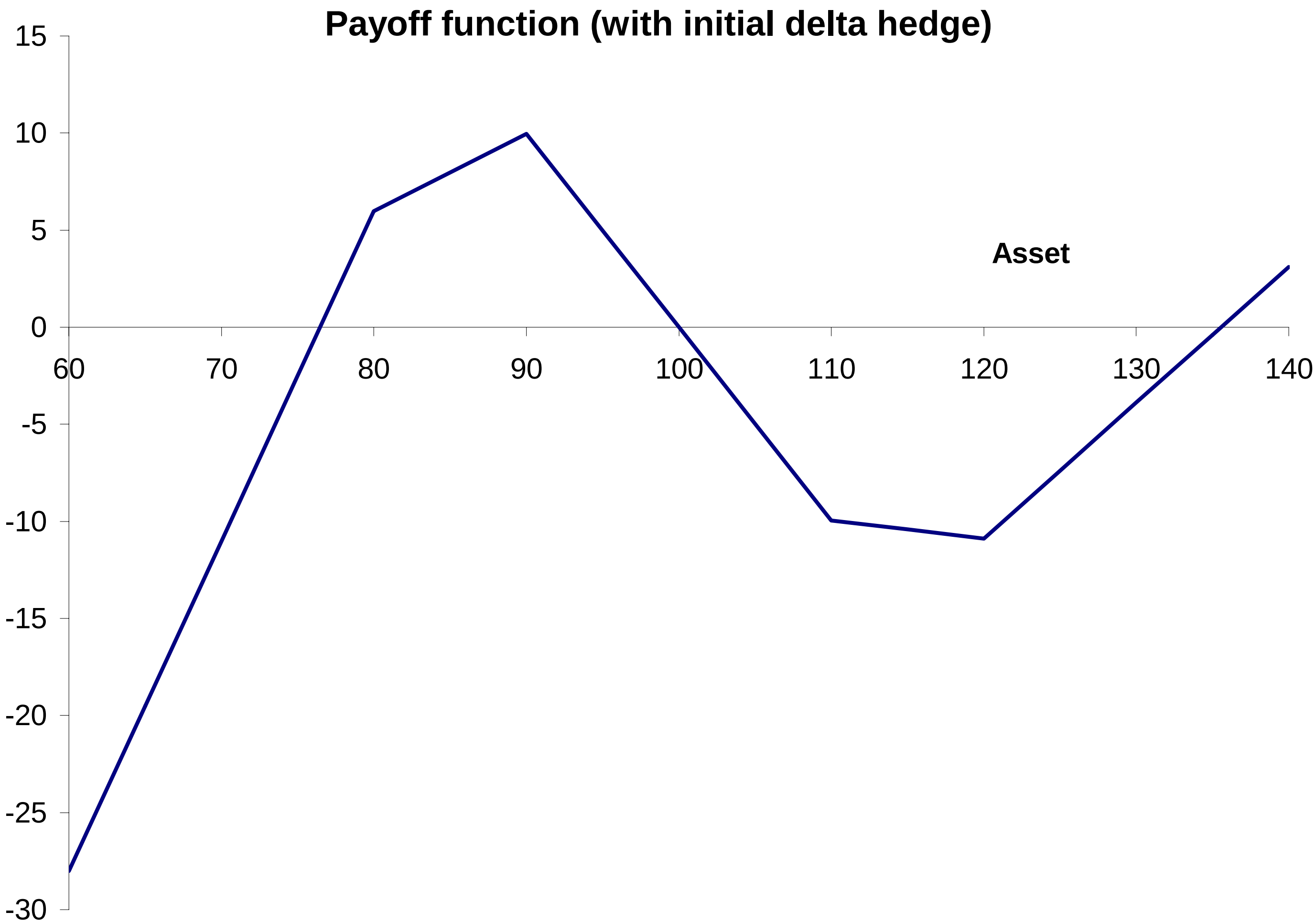
| | **A** | **B** | **C** | **D** | **E** |
|---|---|---|---|---|---|
| **Type** | Put | Put | Call | Call | Call |
| **Strike** | 80 | 90 | 100 | 110 | 120 |
| **Quantity** | -2.10 | -2.25 | 0 | 1.46 | 1.28 |

An optimal portfolio.

The payoff function (with its initial delta hedge) is shown in the next figure.

• This optimization has effectively found an ideal risk reversal trade.

This portfolio would cost -\$0.46 to set up, i.e. it would bring in premium. The expected profit is \$6.83.




Payoff with initial delta hedge for optimal portfolio; $S = 100$, $\mu = 0$, $\sigma = 0.2$, $r = 0.05$, $D = 0$, $T = 1$. See text for additional parameters and information.

Because the state variable representing the profit, $I$, is not normally distributed a portfolio analysis based on mean and variance is open to criticism.

So now we shall look at other ways of choosing or valuing a portfolio.

## Other optimization strategies

Rather than choose an option or a portfolio based on mean and variance it might be preferable to examine the probability density function for $I$.

The main reason for this is the observation that $I$ is not normally distributed.

Mathematically the problem for the cumulative distribution function for the final profit $I'$ can be written as $C(S_0, 0, t_0; I')$ where $C(S, I, t; I')$ is the solution of

$$\frac{\partial C}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \mu S \frac{\partial C}{\partial S} + \tfrac{1}{2}\sum_k \left(\sigma^2 - \tilde{\sigma}_k^2\right)\ e^{-r(t-t_0)} S^2 \Gamma_k^i \frac{\partial C}{\partial I} = 0,$$

subject to the final condition

$$C(S, I, T_{\mathrm{max}}; I') = \mathcal{H}(\mathcal{I}' - \mathcal{I}),$$

where $T_{\mathrm{max}}$ is the expiration of the longest maturity option and $\mathcal{H}(\cdot)$ is the Heaviside function.

The same equation, with suitable final conditions, can be used to choose or optimize a portfolio of options based on criteria such as the following.

- Maximize probability of making a profit greater than a specified amount or, equivalently, minimize the probability of making less than a specified amount

- Maximize profit at a certain probability threshold, such as 95% (a Value-at-Risk type of optimization, albeit one with no possibility of a loss)

Constraints would typically need to be imposed on these optimization problems, such as having a set budget and/or a limit on number of positions that can be held.

## Exponential utility approach

Rather than relying on means and variances, which could be criticized because we are not working with a Gaussian distribution for $I$, or solving a differential equation in three dimensions, which may be slow, there is another possibility, and one that has neither of these disadvantages.

This is to work within a utility theory framework, in particular using constant absolute risk aversion with utility function

$$-\frac{1}{\eta}\, e^{-\eta I}.$$

The parameter $\eta$ is then a person's absolute risk aversion.

The governing equation for the expected utility, $U$, is then

$$\frac{\partial U}{\partial t}+\tfrac{1}{2}\sigma^2 S^2\frac{\partial^2 U}{\partial S^2}+\mu S\frac{\partial U}{\partial S}+\tfrac{1}{2}\sum_k\left(\sigma^2-\tilde{\sigma}_k^2\right)\,e^{-r(t-t_0)}S^2\Gamma_k^i\,\frac{\partial U}{\partial I}=0,$$

with final condition

$$U(S,I,T_{\mathrm{max}})=-\frac{1}{\eta}\,e^{-\eta I}.$$

where $T_{\mathrm{max}}$ is the expiration of the longest maturity option.

We can look for a solution of the form

$$U(S, I, t) = -\frac{1}{\eta} e^{-\eta I} Q(S, t),$$

so that

$$\frac{\partial Q}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 Q}{\partial S^2} + \mu S \frac{\partial Q}{\partial S} - \frac{\eta Q}{2} \sum_k \left(\sigma^2 - \tilde{\sigma}_k^2\right)\, e^{-r(t-t_0)} S^2 \Gamma_k^i = 0,$$

with final condition

$$Q(S, T_{\text{max}}) = 1.$$

Being only a two-dimensional equation this will be very quick to solve numerically. One can then pose and solve various optimal portfolio problems. We shall not pursue this in this lecture.

## Conclusions and further work

This lecture has expanded on the work of Carr and Henrard in terms of final formulas for the statistical properties of the profit to be made hedging mispriced options. We have also indicated how more sophisticated portfolio construction techniques can be applied to this problem relatively straightforwardly. We have concentrated on the case of hedging using deltas based on implied volatilities because this is the most common in practice, giving mark-to-market profit the smoothest behavior. The analysis can be readily extended to hedging using arbitrary $\sigma_h$ with little extra effort. This also opens up further optimization possibilities.

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## Appendix: Derivation of results

### Preliminary results

In the following derivations we often require the following simple results.

First,

$$\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}. \tag{6}$$

Second, the solution of

$$\frac{\partial w}{\partial \tau} = \frac{\partial^2 w}{\partial x^2} + f(x, \tau)$$

that is initially zero and is zero at plus and minus infinity is

$$\frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} \int_{0}^{\tau} \frac{f(x', \tau')}{\sqrt{\tau - \tau'}} e^{-(x-x')^2/4(\tau-\tau')} d\tau' dx'. \tag{7}$$

Finally, the transformations

$$x = \ln(S/E) + \frac{2}{\sigma^2}\left(\mu - \tfrac{1}{2}\sigma^2\right)\tau \quad \text{and} \quad \tau = \frac{\sigma^2}{2}(T - t),$$

turn the operator

$$\frac{\partial}{\partial t} + \tfrac{1}{2}\sigma^2 \frac{\partial^2}{\partial S^2} + \mu S \frac{\partial}{\partial S}$$

into

$$\tfrac{1}{2}\sigma^2 \left( -\frac{\partial}{\partial \tau} + \frac{\partial^2}{\partial x^2} \right). \tag{8}$$

**Result 1: Expectation, single option**

The equation to be solved for $F(S,t)$ is

$$\frac{\partial F}{\partial t}+\tfrac{1}{2}\sigma^2S^2\frac{\partial^2 F}{\partial S^2}+\mu S\frac{\partial F}{\partial S}+\tfrac{1}{2}\left(\sigma^2-\tilde{\sigma}^2\right)\,e^{-r(t-t_0)}S^2\Gamma^i=0,$$

with zero final and boundary conditions. Using the above changes of variables this becomes $F(S,t)=w(x,\tau)$ where

$$\frac{\partial w}{\partial \tau}=\frac{\partial^2 w}{\partial x^2}+\frac{E\left(\sigma^2-\tilde{\sigma}^2\right)e^{-r(T-t_0)}e^{-d_2^2/2}}{\sigma\tilde{\sigma}\sqrt{\pi\tau}}$$

where

$$d_2=\frac{\sigma}{\tilde{\sigma}}\frac{x-\frac{2}{\sigma^2}(\mu-\frac{1}{2}\sigma^2)\tau+\frac{2}{\sigma^2}(r-D-\frac{1}{2}\tilde{\sigma}^2)\tau}{\sqrt{2\tau}}.$$

The solution of this problem is, using (7),

$$\frac{1}{2\pi}\frac{E\left(\sigma^2-\tilde{\sigma}^2\right)e^{-r(T-t_0)}}{\sigma\tilde{\sigma}}\int_{-\infty}^{\infty}\int_0^{\tau}\frac{1}{\sqrt{\tau'}}\frac{1}{\sqrt{\tau-\tau'}}$$

$$\exp\left(-\frac{(x-x')^2}{4(\tau-\tau')}-\frac{\sigma^2}{4\tilde{\sigma}^2\tau'}\left(x'-\frac{2}{\sigma^2}(\mu-\tfrac{1}{2}\sigma^2)\tau'+\frac{2}{\sigma^2}(r-D-\tfrac{1}{2}\tilde{\sigma}^2)\tau'\right)^2\right)\,d\tau'dx'.$$

We write the argument of the exponential function as

$$-a(x'+b)^2+c.$$

And so we have the solution

$$\frac{1}{2\pi}\frac{E\left(\sigma^2-\tilde{\sigma}^2\right)e^{-r(T-t_0)}}{\sigma\tilde{\sigma}}\int_0^{\tau}\frac{1}{\sqrt{\tau'}}\frac{1}{\sqrt{\tau-\tau'}}\int_{-\infty}^{\infty}\exp\left(-a(x'+b)^2+c\right)\,dx'd\tau'$$

$$=\frac{1}{2\sqrt{\pi}}\frac{E\left(\sigma^2-\tilde{\sigma}^2\right)e^{-r(T-t_0)}}{\sigma\tilde{\sigma}}\int_0^{\tau}\frac{1}{\sqrt{\tau'}}\frac{1}{\sqrt{\tau-\tau'}}\frac{1}{\sqrt{a}}\exp\left(c\right)\,d\tau'.$$

It is easy to show that

$$a=\frac{1}{4(\tau-\tau')}+\frac{\sigma^2}{4\tilde{\sigma}^2\tau'}\quad\text{and}\quad c=-\frac{\sigma^2}{4\tilde{\sigma}^2\tau'(\tau-\tau')}\frac{\left(x-\frac{2\tau'}{\sigma^2}(\mu-\frac{1}{2}\sigma^2-r+D+\frac{1}{2}\tilde{\sigma}^2)\right)^2}{\frac{1}{\tau-\tau'}+\frac{\sigma^2}{\tilde{\sigma}^2\tau'}}.$$

With

$$s-t=\frac{2}{\sigma^2}\tau'$$

we have

$$c=-\frac{\left(\ln(S/E)+\left(\mu-\frac{1}{2}\sigma^2\right)(s-t)+\left(r-D-\frac{1}{2}\tilde{\sigma}^2\right)(T-s)\right)^2}{2(\sigma^2(s-t)+\tilde{\sigma}^2(T-s))}.$$

From this follows Result 1, that the expected profit initially ($t=t_0$, $S=S_0$, $I=0$) is

$$\frac{Ee^{-r(T-t_0)}(\sigma^2-\tilde{\sigma}^2)}{2\sqrt{2\pi}}\int_{t_0}^{T}\frac{1}{\sqrt{\sigma^2(s-t_0)+\tilde{\sigma}^2(T-s)}}$$

$$\exp\left(-\frac{\left(\ln(S_0/E)+\left(\mu-\frac{1}{2}\sigma^2\right)(s-t_0)+\left(r-D-\frac{1}{2}\tilde{\sigma}^2\right)(T-s)\right)^2}{2(\sigma^2(s-t_0)+\tilde{\sigma}^2(T-s))}\right)\,ds.$$

**Result 2: Variance, single option**

The problem for the expectation of the square of the profit is

$$\frac{\partial v}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 v}{\partial S^2} + \mu S \frac{\partial v}{\partial S} + \tfrac{1}{2}\left(\sigma^2 - \tilde{\sigma}^2\right)\, e^{-r(t-t_0)} S^2 \Gamma^i \,\frac{\partial v}{\partial I} = 0, \qquad (9)$$

with

$$v(S, I, T) = I^2.$$

A solution can be found of the form

$$v(S, I, t) = I^2 + 2I\, H(S, t) + G(S, t).$$

Substituting this into Equation (9) leads to the following equations for $H$ and $G$ (both to have zero final and boundary conditions):

$$\frac{\partial H}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 H}{\partial S^2} + \mu S \frac{\partial H}{\partial S} + \tfrac{1}{2}\left(\sigma^2 - \tilde{\sigma}^2\right)\, e^{-r(t-t_0)} S^2 \Gamma^i = 0;$$

$$\frac{\partial G}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 G}{\partial S^2} + \mu S \frac{\partial G}{\partial S} + \left(\sigma^2 - \tilde{\sigma}^2\right)\, e^{-r(t-t_0)} S^2 \Gamma^i H = 0.$$

Comparing the equations for $H$ and the earlier $F$ we can see that

$$H = F = \frac{E e^{-r(T-t_0)}(\sigma^2 - \tilde{\sigma}^2)}{2\sqrt{2\pi}} \int_t^T \frac{1}{\sqrt{\sigma^2(s-t) + \tilde{\sigma}^2(T-s)}}$$

$$\exp\left( - \frac{\left(\ln(S/E) + \left(\mu - \tfrac{1}{2}\sigma^2\right)(s-t) + \left(r - D - \tfrac{1}{2}\tilde{\sigma}^2\right)(T-s)\right)^2}{2(\sigma^2(s-t) + \tilde{\sigma}^2(T-s))} \right)\, ds.$$

Notice in this that the expression is a present value at time $t = t_0$, hence the $e^{-r(T-t_0)}$ term at the front. The rest of the terms in this must be kept as the running variables $S$ and $t$.

Returning to variables $x$ and $\tau$, the governing equation for $G(S,t) = w(x,\tau)$ is

$$\frac{\partial w}{\partial \tau} = \frac{\partial^2 w}{\partial x^2} +$$

$$\frac{2}{\sigma^2}\frac{E\sigma\left(\sigma^2-\tilde{\sigma}^2\right)e^{-r(T-t_0)}e^{-d_2^2/2}}{4\tilde{\sigma}\sqrt{\pi\tau}}\frac{E\left(\sigma^2-\tilde{\sigma}^2\right)e^{-r(T-t_0)}}{2\sigma\tilde{\sigma}\sqrt{\pi}}\int_0^\tau \frac{1}{\sqrt{\tau'}}\frac{1}{\sqrt{\tau-\tau'}}\frac{1}{\sqrt{a}}\exp\left(c\right)\,d\tau' \qquad (10)$$

where

$$d_2 = \frac{\sigma}{\tilde{\sigma}}\frac{x-\frac{2}{\sigma^2}(\mu-\frac{1}{2}\sigma^2+r-D-\frac{1}{2}\tilde{\sigma}^2)\tau}{\sqrt{2\tau}},$$

and $a$ and $c$ are as above.

The solution is therefore

$$\frac{1}{2\sqrt{\pi}}\frac{2}{\sigma^2}\frac{E\sigma\left(\sigma^2-\tilde{\sigma}^2\right)e^{-r(T-t_0)}}{4\tilde{\sigma}\sqrt{\pi}}\frac{E\left(\sigma^2-\tilde{\sigma}^2\right)e^{-r(T-t_0)}}{2\sigma\tilde{\sigma}\sqrt{\pi}}\int_{-\infty}^{\infty}\int_0^\tau \frac{f(x',\tau')e^{-d_2^2/2}}{\sqrt{\tau-\tau'}}e^{-(x-x')^2/4(\tau-\tau')}d\tau'dx'.$$

where now

$$f(x',\tau') = \frac{1}{\sqrt{\tau'}}\int_0^{\tau'}\frac{1}{\sqrt{\tau''}}\frac{1}{\sqrt{\tau'-\tau''}}\frac{1}{\sqrt{a}}\exp\left(c\right)\,d\tau''$$

and in $a$ and $c$ all $\tau$s become $\tau'$s and all $\tau'$s become $\tau''$s, and in $d_2$ all $\tau$s become $\tau'$s and all $x$s become $x'$s.

The coefficient in front of the integral signs simplifies to

$$\frac{1}{8\pi^{3/2}}\frac{E^2\left(\sigma^2-\tilde{\sigma}^2\right)^2 e^{-2r(T-t_0)}}{\sigma^2\tilde{\sigma}^2}.$$

The integral term is of the form

$$\int_{-\infty}^{\infty}\int_0^{\tau}\int_0^{\tau'}\cdots d\tau'' d\tau' dx',$$

with the integrand being the product of an algebraic term

$$\frac{1}{\sqrt{\tau'}\sqrt{\tau''}\sqrt{\tau-\tau'}\sqrt{\tau'-\tau''}\sqrt{a}}$$

and an exponential term

$$\exp\left(-\tfrac{1}{2}d_2^2-\frac{(x-x')^2}{4(\tau-\tau')}+c\right).$$

This exponent is, in full,

$$-\frac{1}{4\tau'}\frac{\sigma^2}{\tilde{\sigma}^2}\left(x'-\frac{2}{\sigma^2}(\mu-\tfrac{1}{2}\sigma^2)\tau'+\frac{2}{\sigma^2}(r-D-\tfrac{1}{2}\tilde{\sigma}^2)\tau'\right)^2-\frac{(x-x')^2}{4(\tau-\tau')}$$

$$-\frac{\sigma^2}{4\tilde{\sigma}^2\tau''(\tau'-\tau'')}\frac{\left(x'-\frac{2\tau''}{\sigma^2}(\mu-\frac{1}{2}\sigma^2-r+D+\frac{1}{2}\tilde{\sigma}^2)\right)^2}{\frac{1}{\tau'-\tau''}+\frac{\sigma^2}{\tilde{\sigma}^2\tau''}}.$$

This can be written in the form

$$-d(x'+f)^2+g,$$

where

$$d=\frac{1}{4}\frac{\sigma^2}{\tilde\sigma^2}\frac{1}{\tau'}+\frac{1}{4}\frac{1}{\tau-\tau'}+\frac{1}{4}\frac{\sigma^2}{\sigma^2(\tau'-\tau'')+\tilde\sigma^2\tau''}$$

and

$$g=-\frac{\sigma^2}{4\tilde\sigma^2\tau'}\left(x-\frac{2\alpha\tau'}{\sigma^2}\right)^2-\frac{\sigma^2}{4(\sigma^2(\tau'-\tau'')+\tilde\sigma^2\tau'')}\left(x-\frac{2\alpha\tau''}{\sigma^2}\right)^2$$

$$+\frac{1}{4}\frac{\left(\frac{\sigma^2}{\tilde\sigma^2\tau'}\left(x-\frac{2\alpha\tau'}{\sigma^2}\right)+\frac{\sigma^2}{(\sigma^2(\tau'-\tau'')+\tilde\sigma^2\tau'')}\left(x-\frac{2\alpha\tau''}{\sigma^2}\right)\right)^2}{\frac{\sigma^2}{\tilde\sigma^2}\frac{1}{\tau'}+\frac{1}{\tau-\tau'}+\frac{\sigma^2}{\sigma^2(\tau'-\tau'')+\tilde\sigma^2\tau''}},$$

$$\alpha=\mu-\tfrac{1}{2}\sigma^2-r+D+\tfrac{1}{2}\tilde\sigma^2.$$

Using Equation (6) we end up with

$$\frac{1}{4\pi^{3/2}}\frac{E^2\left(\sigma^2-\tilde\sigma^2\right)^2e^{-2r(T-t_0)}}{\sigma^2\tilde\sigma^2}$$

$$\int_0^\tau\int_0^{\tau'}\frac{1}{\sqrt{\tau'}\sqrt{\tau''}\sqrt{\tau-\tau'}\sqrt{\tau'-\tau''}\sqrt{a}}\sqrt{\frac{\pi}{d}}\exp(g)\,d\tau''d\tau'.$$

Changing variables to

$$\tau=\frac{\sigma^2}{2}(T-t),\quad \tau'=\frac{\sigma^2}{2}(T-s),\quad \text{and}\quad \tau''=\frac{\sigma^2}{2}(T-u),$$

and evaluating at $S=S_0$, $t=t_0$, gives the required Result 2.

**Result 3: Expectation, portfolio of options**

This expression follows from the additivity of expectations.

**Result 4: Variance, portfolio of options**

The manipulations and calculations required for the analysis of the portfolio variance are similar to that for a single contract. There is again a solution of the form

$$v(S, I, t) = I^2 + 2I\ H(S, t) + G(S, t).$$

The main differences are that we have to carry around two implied volatilities, $\tilde{\sigma}_j$ and $\tilde{\sigma}_k$ and two expirations, $T_j$ and $T_k$. We will find that the solution for the variance is the sum of terms satisfying diffusion equations with source terms like in Equation (10). The subscript '$k$' is then associated with the gamma term, and so appears outside the integral in the equivalent of (10), and the subscript '$j$' is associated with the integral and so appears in the integrand.

There is one additional subtlety in the derivations and that concerns the expirations. We must consider the general case $T_j \neq T_k$. The integrations in (5) must only be taken over the intervals up until the options have expired. The easiest way to apply this is to use the convention that the gammas are zero after expiration. For this reason the $s$ integral is over $t_0$ to $\min(T_j, T_k)$.

 